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THE

Young Mathematician's Guide:

Being a PLAIN and EASY

INTRODUCTION

TO THE

Mathematicks.

In Five Parts.

VIZ.

I. **Arithmetick,** Vulgar and Decimal, with all the useful Rules; And a General Method of Extracting the Roots of all Single Powers.

II. **Algebra,** or Arithmetick in Species; wherein the Method of Raising and Resolving Æquations is rendered Easy; and Illustrated with Variety of Examples, and Numerical Questions. Also the whole Business of Interest and Annuities, *&c.* perform'd by the Pen.

III. The **Elements** of **Geometry,** Contracted, and Analytically Demonstrated; With a New and Easy Method of finding the Circle's Periphery and Area to any assigned Exactness, by one Æquation only; Also a New Way of making Sines and Tangents.

IV. **Conick-Sections,** wherein the Chief Properties, *&c.* of the Ellipsis, Parabola, and Hyperbola, are clearly Demonstrated.

V. The **Arithmetick** of **Infinites** Explain'd, and render'd Easy; with its Application to superficial and solid Geometry.

With an APPENDIX of **Practical Gauging.**

The FIFTH EDITION, *carefully Corrected; and New Tables of Compound Interest at Five* per Cent. *Calculated, and Added by the Author,*

JOHN WARD.

LONDON: Printed for A. BETTESWORTH, at the *Red-Lyon* in *Paternoster-row*; and F. FAYRAM, at the *South-Entrance* of the *Royal Exchange.* 1728.

To the Honourable

Sir Richard Grosvenor of *Eaton*, in the County *Palatine* of *Cheſter*, Baronet.

SIR,

WHEN requeſted by ſome Bookſellers in *London*, to Reviſe and Prepare this Treatiſe for a New Impreſſion, and once reſolved to Anſwer their Demands; I was not long conſidering at whoſe Feet to lay it.

My Memory may indeed be impair'd by Age, Misfortunes and Accident; nay, I am ſenſible it is ſo: But it muſt be entirely loſt, when I am forgetful of the great Obligations I lie under to Sir *Richard Groſvenor*.

Your Hoſpitality and Generoſity make you ſtand unenvied in the abundance of Fortune. Any Upſtart may contrive to ſpend a Great Eſtate; But it is a Felicity almoſt peculiar to Great Birth to become One.

Were I now to deſcribe Liberality without Profuſeneſs; Steadineſs in Principles, without any private View; Candor and Affability, Good Nature join'd to ſound Judgment, and a Serenity of Temper, which your Enemies will always find the Companion of true Courage; And then pronounce that you

are poſſeſſed of all theſe good Qualities in as high a Degree as moſt Men living; No Gentleman that knows you well, would think I flatter'd you.

Sir, Give me Leave to ſay, I Honour your Character, and Love your Perſon; My Expreſſions are uncourtly, my Stile unpoliſh'd, and therefore more proper to be prefix'd to a Work wherein the Matters related are indeed clad in a plain and homely Dreſs; but they are True, and deſigned to propagate Mathematical Learning amongſt ſuch as deſire to be introduced into that ſort of Knowledge; And I am extreamly pleas'd they are permitted to be ſent into the World under your Protection.

That you may long Live, to promote the Good of your Country, and that City in whoſe Intereſt you have ſo heartily engag'd your Self; And that you may ever ſucceed in your own private Affairs, and live to enjoy all the Bleſſings that attend a quiet prudent Life, is the earneſt Prayer of,

Honoured SIR,

Your moſt Obliged, Humble,

and Obedient Servant,

J. WARD.

The PREFACE.

To the READER.

I *Think it Needleſs* (and almoſt Endleſs) *to run over all the Uſefulneſs, and Advantages of* Mathematicks *in General; ſhall therefore only touch upon thoſe Two Admirable Sciences,* ARITHMETICK *and* GEOMETRY; *which are indeed the Two Grand Pillars* (or rather the Foundations) *upon which all other Parts of Mathematical Learning depend.*

As to the Uſefulneſs of Arithmetick, *'Tis well known that no Buſineſs, Commerce, Trade, or Imployment whatſoever, even from the Merchant to the Shop-keeper,* &c. *can be manag'd and carry'd on, without the Aſſiſtance of Numbers.*

And as to the Uſefulneſs of Geometry, *'Tis as certain, that no curious Art, or Mechanick-Work, can either be invented, improved or performed, without its aſſiſting Principles; tho' perhaps the Artiſt, or Workman, has but little* (nay, ſcarce any) *Knowledge in* Geometry.

Then, as to the Advantages that ariſe from both theſe Noble Sciences, when duly join'd together, to aſſiſt each other, and then Apply'd to Practice (according as Occaſion requires) *will readily be granted by all who conſider the vaſt Advantages that Accrue to Mankind from the Buſineſs of Navigation only. As alſo from that of Surveying and Dividing of Lands betwixt Party and Party. Beſides the great Pleaſure and Uſe there is from Time-keepers, as Dials, Clocks, Watches,* &c. *All theſe, and a great many more very uſeful Arts,* (too many to be enumerated here) *wholly depend upon the aforeſaid Sciences.*

And therefore 'tis no Wonder, That in all Ages ſo many Ingenious and Learned Perſons have imploy'd themſelves in writing upon the Subject of Mathematicks; *but then moſt of thoſe Authors ſeem to preſuppoſe that their Readers had made ſome Progreſs in that ſort of Learning before they attempted to peruſe thoſe Books, which are generally Large Volumns, written in ſuch abſtruſe Terms that young Learners were really afraid of looking into thoſe Studies.*

Theſe Conſiderations firſt put me (many Years ago) *upon the Thoughts of Endeavouring to Compoſe ſuch a plain and familiar Introduction to the* Mathematicks, *as might Encourage thoſe that were willing* (to ſpend ſome Time that way) *to venture and proceed on with Chearfulneſs; Tho' perhaps they were*

wholly

wholly ignorant of its first Rudiments. Therefore I began with their first Elements or Principles.

That is, I began with an Unit in Arithmetick, and a Point in Geometry; And from these Foundations proceeded gradually on, leading the young Learner Step by Step with all the Plainness I can, &c.

*And for that Reason I published this Treatise (*Anno 1707*) by the Title of the* Young Mathematician's Guide; *which has Answer'd the Title so well, that I believe I may truly say* (without Vanity) *this Treatise hath prov'd a very helpful Guide to near five thousand Persons; and perhaps most of them such as would never have look'd into the* Mathematicks *at all but for it.*

And not only so, but it hath been very well received amongst the Learned, and (I've been often told) *so well Approv'd on at the Universities, in* England, Scotland, *and* Ireland, *that it's Order'd to be publickly read to their Pupils,* &c.

The Title Page gives a short Account of the several Parts treated of, with the Corrections and Additions that are made to this Fifth Edition, *which I shall not inlarge upon, but leave the* Book *to speak for it self; and if it be not able to give Satisfaction to the Reader, I'm sure all I can say here in its behalf will never recommend it: But this may be truly said, That whoever reads it over, will find more in it than the Title doth promise, or perhaps he expects: 'Tis true indeed, the Dress is but Plain and Homely, it being wholly intended to Instruct, and not to Amuse or Puzzle the young Learner with hard Words, and obscure Terms: However, in this I shall always have the Satisfaction; That I've sincerely aim'd at what's useful, tho' in one of the meaneſt ways; 'Tis Honour enough for me to be accounted as one of the Under-Labourers in Clearing the Ground a little, and removing some of the Rubbish that lay in the way to this sort of Knowledge. How well I have perform'd That, must be left to proper Judges.*

To be brief; As I am not sensible of any Fundamental Error in this Treatise, so I will not pretend to say it is without Imperfections, (Humanum est errare) *which I hope the Reader will excuse, and pass over with the like Candor and Good Will that it was composed for his Use; by his real Well-wisher,*

J. WARD.

London, October 10th, 1706

Corrected, &c. at Chester,
January 20th, 1722.

THE

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AN

AN INTRODUCTION TO THE Mathematicks.

PART I.

Præcognita.

THE *Buſineſs of* MATHEMATICKS *in all its Parts, both* Theory *and* Practice, *is only to ſearch out and determine the true Quantity; either of* Matter, Space, *or* Motion, *according as Occaſion requires.*

By Quantity *of* Matter *is here meant the* Magnitude *or Bigneſs of any viſible thing, whoſe* Length, Breadth *and* Thickneſs *may either be meaſured, or eſtimated.*

By Quantity *of* Space *is meant the diſtance of one thing from another.*

And by Quantity *of* Motion *is meant the ſwiftneſs of any thing moving from one place to another.*

The conſideration of theſe, according as they may be propoſed, are the Subjects *of the* Mathematicks, *but chiefly that of* Matter.

Now the conſideration of Matter, *with reſpect to its* Quantity, Form *and* Poſition, *which may either be* Natural, Accidental, *or* Deſigned, *will admit of infinite Varieties; But all the Varieties that are yet known, or indeed poſſible to be conceived, are wholly compriſed under the due conſideration of theſe Two,* Magnitude *and* Number, *which are the proper Subjects of* Geometry, Arithmetick *and* Algebra. *All other* Parts *of the* Mathematicks *being only the Branches of theſe three* Sciences, *or rather their Application to particular Caſes.*

Geometry *is a* Science *by which we ſearch out and come to know either the whole* Magnitude, *or ſome part of any propoſed* Quantity; *and is to be obtained by comparing it with another known* Quantity *of the ſame kind, which will always be one of theſe,* viz. *A* **Line** *(or* Length *only) A* **Surface**, *(that is,* Length *and* Breadth) *or a* **Solid** *(which hath* Length, Breadth *and* Depth, *or* Thickneſs) *Nature admitting of no other Dimenſions but theſe Three.*

Arithmetick *is a* Science *by which we come to know what* Number *of* Quantities *there are (either real or imaginary) of any kind, contained in another* Quantity *of the ſame kind: Now this Conſideration is very different from that of* Geometry, *which is only to find out true and proper Anſwers to all ſuch Queſtions as demand, how* Long, *how* Broad, *how* Big, *&c. But when we are to conſider either of more* Quantities *than* one, *or how often one* Quantity *is contained in another, then we have recourſe to* Arithmetick, *which is to find out true and proper Anſwers to all ſuch Queſtions as demand* how Many, *what* Number, *or* Multitude *of* Quantities *there are. To be brief, the Subject of* Geometry *is that of* Quantity, *with reſpect to its* Magnitude *only; and the Subject of* Arithmetick *is* Quantities *with reſpect to their* Number *only.*

Algebra *is a* Science *by which the moſt abſtruſe or difficult Problems either in* Arithmetick *or* Geometry *are Reſolved and Demonſtrated, that is, it equally interferes with them both; and therefore its promiſcuouſly named, being ſometimes called* Specious Arithmetick, *as by* Harriot, Vieta, *and Doctor* Wallis, *&c. And ſometimes its called* Modern Geometry, *particularly the ingenious and great Mathematician, Mr.* Edmund Halley, *Savilian Profeſſor of* Geometry *in the Univerſity of* Oxford, *giving this following Inſtance of the Excellence of our* Modern Algebra, *writes thus,*

'*The Excellence of the* Modern Geometry (*ſaith he*) *is in* 'nothing more evident, *than in thoſe full and* Adequate Solutions 'it gives *to* Problems: *Repreſenting all the poſſible Caſes at one* 'view, *and in one general* Theorem *many times comprehending* 'whole Sciences; *which deduced at length into* Propoſitions *and* 'demonſtrated *after the manner of the* Ancients, *might well* 'become *the Subjects of large Treatiſes: For whatſoever* Theorem 'ſolves *the moſt complicated* Problem *of the kind, does with a* 'due Reduction *reach all the Subordinate* Caſes.' *Of which he gives a notable Inſtance in the* Doctrine *of* Dioptricks *for finding the* Foci *of* Optick Glaſſes *univerſally, (vide* Philoſophical Tranſactions, *Numb.* 205.)

Thus

Thus you have a ſhort and general Account of the proper Subjects of thoſe three Noble and Uſeful Sciences, Arithmetick, Geometry *and* Algebra. *I ſhall now proceed to give a particular Account of each, and firſt of* Arithmetick, *which is the* Baſis *or* Foundation *of all* Arts, *both* Mathematick *and* Mechanick; *and therefore it ought to be well underſtood before the reſt are medled withal.*

CHAP. I.

Concerning the ſeveral Parts of **Arithmetick**, *with the Definition of ſuch Characters as are uſed in this Treatiſe.*

Arithmetick, or the *Art of Numbring*, is fitly divided into three diſtinct Parts, two of which are properly called *Natural*, and the third *Artificial*.

The firſt being the moſt plain and eaſieſt, is commonly called *Vulgar Arithmetick* in whole *Numbers*; becauſe every *Unit* or *Integer* concerned in it, repreſents one whole *Quantity* of ſome *Species* or thing propoſed.

The ſecond is that which ſuppoſes an *Unit* (and conſequently the *Quantity* or thing repreſented by that *Unit*) to be *Broken* or *Divided* into equal *Parts* (either even or uneven) and conſiders of them either as pure *Parts*, viz. Each leſs than an *Unit*, or elſe of *Parts* and *Integers* intermixt. And is uſually called the *Doctrine of Vulgar Fractions*.

The third, or *Artificial Part*, is called *Decimal Arithmetick*; being an *Artificial* Invention of managing *Fractions* or *Broken Numbers*, by a much more commodious and eaſy way than that of *Vulgar Fractions*: For the ſeveral Operations performed in *Decimals*, differ but little from thoſe in *Whole Numbers*; and therefore it is now become of general Uſe, eſpecially in *Geometrical Computations*.

Arithmetick (in all its *Parts*) is performed by the various ordering and diſpoſing of Ten *Arabick Characters* or *Numeral Figures* (which by ſome are called *Digits*)

viz. { *One Two Three Four Five Six Seven Eight Nine Cypher*
1 2 3 4 5 6 7 8 9 0

The uſe of theſe Characters *is ſaid to be firſt introduced into* England *near ſix hundred Years ago*, viz. *about the Year* 1130, *vide Doctor* Wallis's *Algebra*, Page 12.

The first of these *Characters* is called *Unity*, and represents one of any kind of *Species* or *Quantity*. As one *World*, one *Star*, one *Man*, &c.

Viz. *Unity* is that by which every thing that is, is called one (*Euclid.* 7. *Def.* 1.) and is the beginning of all *Numbers*. That is to say, *Number* is a *Multitude* of *Units*. Euclid. 7. Def. 2.

For, one more one, makes Two; and one, more one, more one makes Three, &c. *Which is the first and chief* Postulate, *or Rather* Axiom *to* Arithmetick.

Viz. { That 1+1=2. 1+1+1=3. 1+1+1+1=4. 1+1+1+1+1=5. And so on to 9.

Nine of these *Figures* were thus composed of *Units*, and differently form'd to represent so many *Units* put together into one *Sum*, as was intended each should denote: *Nine* being the greatest *Number* of *Units* that was then thought convenient to be expressed by one single *Character*; the last of the *Ten* is only a *Cypher*, or (as some phrase it) a Nothing, because of it self it signifies nothing; for if never so many *Cyphers* be added to, or *Substracted* from, any *Number*, they can neither increase nor diminish that *Number*; but yet as a *Cypher* (or *Cyphers*) may be placed, the other *Figures* will become of different *Values* from what they were before, as will appear further on.

For the more convenient ordering of the aforesaid *Numeral Figures*, according to the several Varieties that happen in *Computations*; I do advise the young Learner to acquaint himself with the Signification of the following *Algebraick Signs* or *Characters*, which he will find of excellent Use, as being a much shorter, better and more significant way of denoting what is to be done (in most Operations) than can otherwise be expressed in Words at length.

Significations.

Signs *Names.*

\+ } { *Plus* or *more.* — The Sign of *Addition*; As 8+7 is 8 more 7, and signifies that the *Numbers* 8 and 7 are to be Added into one *Sum*. The like is to be understood when several *Numbers* are connected together with the Sign +

As 34+22+9+45, &c. denotes these are all to be added into one *Sum*.

The

— } { *Minus or leſs.* } The Sign of *Subſtraction*; As 9—6 is 9 leſs 6, and ſignifies that 6 is to be taken from 9, that ſo their Difference may be found.

× } { *Into or with.* } The Sign of *Multiplication*; as 9×6, is 9 into 6, and ſignifies that 9 is to be Multiplied into or with 6.

÷ } { *By.* } The Sign of *Diviſion*; as 8÷2, is 8 by 2, and ſignifies that 8 is to be Divided by 2, alſo thus 2) 8 (4. or thus $\frac{8}{2}$ each ſignifying the ſame thing, to wit, 8 Divided by 2.

= } { *Equal.* } The Sign of *Equality* or *Equation*, viz. whenever this Sign = is placed betwixt *Numbers* (or *Quantities*) it denotes them to be Equal, as 9=9, or 9+6=15, or 9—6=3, *&c.* That is, 9 is Equal to 9, or 9 more 6 is Equal to 15, and 9 leſs 6 is Equal to 3, *&c.*

: : } { *So is.* } The Sign of *Proportion*, or that commonly called the *Golden Rule*, or *Rule of Three*, and : : is always placed betwixt the Two middle *Terms* or *Numbers* in Proportion. Thus, 2 : 8 : : 6 : 24. To be read thus; As 2, Is to 8, So is 6, To 24.

Theſe *Signs* and their *Significations*, being perfectly learnt, will help to ſhorten the Work.

CHAP. II.

Concerning the Principal Rules *in* Arithmetick, *and how they are performed in Whole Numbers.*

THE *Rules* by which *Numerical Operations* are perform'd in all the Parts of *Arithmetick*, are many and various, ſeveral of them being form'd and raiſed as Occaſion requires, when applied to *Practice*, yet they are all comprehended within the due Conſideration of theſe Six, *viz.* Numeration (or Notation) Addition,

Addition, Substraction, Multiplication, Division, and Evolution, or *Extraction of Roots.*

Sect. 1. *Of* Numeration *or* Notation.

Numeration or *Notation*, teacheth to Read or Expreſs the true Value of any *Number* when writ down; and conſequently to write down any propoſed *Number* according to its true Value when it is named: And this conſiſteth of Two parts;

1. The due order of placing down Figures.
2. The true valuing of each Figure in its place.

Both which are plainly exhibited in the following Table.

&c.	Hundreds of Thouſands of Millions	Tens of Thouſands of Millions	Thouſands of Millions	Hundreds of Millions	Tens of Millions	Millions	Hundreds of Thouſands	Tens of Thouſands	Thouſands	Hundreds	Tens	Units { And is the firſt place in Counting.
	6	7	8	9	8	7	6	5	4	3	2	1
	Period of Thouſands of Millions			Period of Millions			Period of Thou-ſands			Period of Units		

By this **Numeration** *Table* it's apparent, that the order of Places is reckoned from the Right-Hand towards the Left; the Firſt place of any *Number* being always that which is the outmoſt *Figure* to the Right-Hand; and whatever *Figure* ſtands in that place, doth only ſignify its own ſimple value, *viz.* ſo many *Units* as that *Figure* repreſents.

The ſecond place is that of *Tens*, any *Figure* ſtanding in that place ſignifieth ſo many *Tens* as that *Figure* repreſents *Units*.

The

The third place is *Hundreds*, the fourth place *Thouſands*, &c. That is, each place towards the Left-Hand is *Ten* times the value of that next it towards the Right.

For inſtance, ſuppoſe 759 were propoſed to be read or pronounced according to the Value of each *Figure* as they now ſtand. The firſt *Figure* in this *Sum* is 9, becauſe it ſtands in the place of *Units*, and therefore ſignifies but its own ſimple Value, to wit, 9 *Units*, or *Nine*. The ſecond *Figure* 5 ſtands in the place of *Tens*, and therefore ſignifies Five *Tens* or *Fifty*. The *Figure* 7 ſtands in the third place, or place of *Hundreds*, and therefore it ſignifies *Seven Hundred*, and the whole Sum is to be read or pronounced thus, *Seven Hundred Fifty Nine*.

Note, Although the *Figure* 7 ſtands in the third place (according to the order of *Numbering*) yet when the whole *Sum* comes to be read it's firſt pronounced, the reading of *Numbers* being perform'd like that of Letters or Words, always beginning with the outmoſt *Figure* towards the Left-Hand, and ſo many *Figures* as are placed together without any Point, Comma, Line, or other Note of Diſtinction between them, are all but one *Sum*, and muſt be read as ſuch.

For Example, 763596 is but one intire *Sum* or *Number*, notwithſtanding it conſiſts of ſix places of *Figures*, and is thus read; *Seven Hundred Sixty Three Thouſand, Five Hundred Ninety Six.*

The like is to be obſerved in reading or expreſſing the true Value of any *Sum* or *Rank* of *Numbers* conſiſting of *Seven*, *Eight*, *Nine*, or more places of *Figures*, each *Figure* being to be valued according to its diſtance from the place of *Unity*: As in the foregoing Table.

Now ſuch Values may as well ariſe by *Cyphers*, as by other *Figures*; for inſtance, 6 ſtanding by it ſelf, repreſents but Six *Units*: But if a *Cypher* be annext to it thus, 60, then it becomes *Sixty*; for the *Cypher* poſſeſſing the place of *Units*, hath thereby removed the 6 into the place of *Tens*; and another *Cypher* more would make it 600, *Six Hundred*, &c.

Whence it may be noted, that altho' a *Cypher* of it ſelf ſignify nothing (as hath been ſaid before) yet being placed on the Right Hand of any *Figure*, it augments the Value of that *Figure* by advancing it into a higher place than otherwiſe it would have been, had not the *Cypher* been there.

Take one Example more in *Numeration*, if you pleaſe, that in the Table, *viz.* 6789876543 21, which is, according as is there ſignified,

Six

Six Hundred Seventy Eight Thouſand Millions,
Nine Hundred Eighty Seven Millions,
Six Hundred Fifty Four Thouſand,
Three Hundred Twenty One Units. Of any propoſed *Species* or *Quantities* whatſoever.

And here it may be obſerved, that every third *Figure* from the place of *Units*, bears the Name of *Hundreds*; which ſhews that if any great *Sum* be parted, or rather diſtinguiſhed into Periods, of Three *Figures* in each *Period* (as in the foregoing Table) it will be of good uſe to help the young Learner in the eaſier valuing and expreſſing that *Sum*.

Sect. 2. *Of* Addition.

Poſtulate or Petition.

That any given Number *may be increaſed or made more, by putting another* Number *to it.*

Addition is that *Rule* by which ſeveral *Numbers* are collected and put together, that ſo their *Sum* or *Total* Amount may be known.

In this *Rule* Two things being carefully obſerved, the Work will be eaſily performed.

1. The firſt is the true placing of the *Numbers*, ſo as that each *Figure* may ſtand directly underneath thoſe *Figures* of the ſame Value, *viz.* place *Units* under *Units*, *Tens* under *Tens*, and *Hundreds* under *Hundreds*, &c.

Then underneath the loweſt Rank (always) draw a Line to ſeparate the given *Numbers* from their *Sum* when it's found.

Example. If theſe *Numbers* 54327, and 2651, were given to be Added together, they muſt be placed

Thus, { 54327
2651

2. The ſecond thing to be obſerved is the due Collecting or Adding together each Row of *Figures* that ſtand over one another of the ſame Value: and that is thus performed.

Rule.

Always begin your Addition *at the place of* Units, *and Add together all the* Figures *that ſtand in that place, and if their* Sum *be under* Ten, *ſet it down below the Line underneath its own place; but if their* Sum *be more than* Ten, *you muſt ſet down only the overplus, or odd* Figure *above the* Ten (*or* Tens) *and ſo many* Tens *as the* Sum *of thoſe* Units *amount to, you muſt carry*

to the place of Tens; *Adding them and all the* Figures *that ſtand in the place of* Tens *together, in the ſame manner as thoſe of the* Units *were Added; then proceed in the ſame order to the place of* Hundreds, *and ſo on to each place until all is done.*

The *Sum* ariſing from thoſe *Additions* will be the *Total* Amount required.

Example 1.

Let it be required to find the Sum of the aforeſaid *Numbers*, *viz.* { 54327 / 2651

56978 the *Sum* required.

Beginning at the place of *Units*, I ſay 1 and 7 is 8, which being leſs than 10, I ſet it down (according to the *Rule*) underneath its own place of *Units*; and then proceed to the place of *Tens*, ſaying 5 and 2 is 7, which being leſs than 10, I ſet it down underneath its own place of *Tens*, and proceed to do the like at the place of *Hundreds*, and then at *Thouſands*, ſetting each of their *Sums* underneath their own reſpective places: Laſtly, becauſe there is not any *Figure* in the lower Rank to be added to the *Figure* 5, which ſtands in the place of *Ten Thouſands*, in the upper Rank, I therefore bring down the ſaid 5 to the reſt, placing it underneath its own place, and then I find that 54327+2651=56978, the true *Sum* required.

Example 2.

Suppoſe it were required to find the *Sum* of theſe *Numbers*, 3578+496+742+184+95. Theſe being placed, as before directed, will ſtand as in the Margin. Then beginning (as before) at the place of *Units*, ſay 5 and 4 is 9, and 2 is 11, and 6 is 17, and 8 is 25; ſet down the 5 *Units* underneath its own place of *Units*, and carry the 20, or two *Tens*, to the place of *Tens* (at which place they are only 2) ſaying, 2 and 9 is 11, and 8 is 19, and 4 is 23, and 9 is 32, and 7 is 39; ſet down the 9 underneath its own place of *Tens*, and carry the 30, or three *Tens* (which indeed is 300) to the place of *Hundreds*, at which place they are but 3, ſaying, 3 I carry and 1 is 4, and 7 is 11, and 4 is 15, and 1 is 20; here becauſe there is no *Figure* overplus (as before) I ſet down a *Cypher* underneath the place of *Hundreds*, and carry the two *Tens* (or rather the 2000) to the place of *Thouſands*, ſaying (as

3578
496
742
184
95
5095

(as before) 2 I carry and 3 is 5, which being the laſt, I ſet it down underneath its own place, and all is finiſhed. And find the *Sum* or *Total* Amount to be 5095=3578+496+742+184+95.

If this Example be well conſidered, it will be ſufficient to ſhew the uſual Method of *Addition* in Whole *Numbers*; but to make all plain and clear, I ſhall ſhew the young Learner the Reaſon of carrying the *Tens* from one Degree or Row of *Figures*, to the next Superior Degree, which is done purely to ſave Trouble, and prevent the uſing of more *Figures* than are really neceſſary, as will appear by the following Method of adding together the ſame *Numbers* of the laſt Example.

Thus, Add together each ſingle Row of *Figures* by it ſelf; as if there were no more but that one Row, ſetting down the *Sum* underneath its own place.

	3578
	496
	742
	184
	95
	5095

The *Sum* of the Row of *Units*, is	25	Add
The *Sum* of the Row of *Tens*, is	370	
The *Sum* of the Row of *Hund.* is	1700	
The three *Thouſand* brought down	3000	
The *Sum* or *Total* Amount as before, is	5095	

From hence I preſume it will be eaſy to conceive the true Reaſon of carrying the aforeſaid *Tens*; and alſo that *Cyphers* do not augment or increaſe the *Sum* in *Addition*. (See *Page* 4.)

I might have here inſerted a Lineal Demonſtration of this *Rule* of *Addition*; but I thought it would rather puzzle than improve a young Learner, eſpecially in this place; beſides the Reaſon of it is ſufficiently evident from that Natural Truth of *the Whole being Equal to all its Parts taken together*. Euclid. 1. Axiom 19.

That is, the *Numbers* which are propoſed to be Added together are by that *Axiom* underſtood to be the ſeveral Parts, and their *Sum* or *Total* Amount found by *Addition* is underſtood to be the Whole.

And from thence is deduced the Method of proving the Truth of any *Operation* in *Addition*, *viz.* By parting or ſeparating the given *Numbers* into Two Parcels (or more, according to the Largeneſs of it) and then Adding up each Parcel by it ſelf: For if thoſe particular *Sums* ſo found, be *Added* into one *Sum*, and that *Sum* prove Equal or the ſame with the *Total Sum* firſt found,

found, then all is Right; if not, care must be taken to discover and correct the Error.

Example.

Add { 5647, 3289, 4016 } The *Sum* of these Parts is, 12952

Add { 2900, 5007, 1606 } The *Sum* of these, is 9513

The *Total Sum* of all these Parts 22465 — The *Sum* of each Parcel put together 22465

Sect. 3. *Of* Subtraction.

Postulate or Petition.

That any Number *may be Diminished, or made Less, by taking another* Number *from it.*

Subtraction is that *Rule* by which one *Number* is *Deducted* or taken out of another, that so the *Remainder*, *Difference*, or *Excess* may be known.

As 6 taken out of 9, there *Remains* 3. This 3 is also the *Difference* betwixt 6 and 9, or it is the *Excess* of 9 above 6.

Therefore the *Number* (or *Sum*) out of which *Subtraction* is required to be made, must be greater than (or at least equal to) the *Subtrahend* or *Number* to be *Subtracted*.

Note, *This Rule is the Converse or Direct contrary to* Addition.

And here the same Caution that was given in *Addition*, of placing *Figures* directly under those of the same Value, *viz. Units* under *Units*, *Tens* under *Tens*, and *Hundreds* under *Hundreds*, &c. Must be carefully observed; also underneath the lowest Rank there must be drawn a Line (as before in *Addition*) to separate the given *Numbers* from their *Difference* when it's found.

Then having placed the lesser *Number* under the greater, the *Operation* may be thus performed.

Rule.

Begin at the Right Hand Figure *or place of* Units (*as in* Addition) *and take or* Subtract *the lower* Figure *in that place from*

from the Figure *that ſtands over it, ſetting down the* Remainder *or* Difference *underneath its own place. If the Two* Figures *chance to be Equal, ſet down a* Cypher : *But if the upper* Figure *be leſs than the lower* Figure, *then you muſt* Add 10 *to the upper* Figure, *or mentally call it* 10 *more than it is, and from that* Sum Subſtract *the lower* Figure, *ſetting down the* Remainder (as before directed). *Now becauſe the* 10 *thus added, was ſuppos'd to be borrowed from the next Superior place* (viz. *of* Tens) *in the upper* Figures, *therefore you muſt either call the upper* Figure *in that place from whence the* 10 *was borrowed, one leſs than really it is, or elſe (which is all one, and moſt uſual) you muſt call the lower* Figure *in that place one more than it really is, and then proceed to* Subſtraction *in that place, as in the former ; and ſo gradually on from one Row of* Figures *to another until all be done.*

Example 1.

Let it be required to find the *Difference* between 6785, and 4572. That is, let 4572 be *Subſtracted* from 6785.

Theſe *Numbers* being placed down, as before directed, will ſtand

Thus { 6785
4572

2213

Beginning at the place of *Units*, take 2 from 5 and there will *Remain* 3, which muſt be ſet down underneath its own place, and then proceed to the place of *Tens*, taking 7 from 8, and there will *Remain* 1, to be ſet down underneath its own place ; again, at the place of *Hundreds*, take 5 from 7, and there *Remains* 2, which ſet down, as before ; laſtly, take 4 from 6 and there will *Remain* 2, which being ſet down underneath its own place, the Work is finiſhed, and the *Difference* ſo found will be 2213=6785—4572, as was required.

Example 2.

The *Difference* between 5849, and 7496 is required.

Having placed the *Numbers* as in the Margin, begin at the place of *Units* (as before) and ſay 9 from 6 cannot be, but 9 from 16 and there *Remains* 7, to be ſet down under its own place ; next proceed to the place of *Tens*, where you muſt now pay the 10 that was borrowed to make the 6, 16, by accounting the upper *Figure* 9 in that place one leſs than it is, ſaying, 4 from 8 and there *Remains* 4, or elſe (which is the moſt practiſed) ſay 1 I borrowed and 4 is 5 from

7496
5849
1647

from 9 and there *Remains* 4, to be ſet down under its own place (as before); again, at the place of *Hundreds*, ſay 8 from 4 that cannot be, but 8 from 14 there will *Remain* 6 to be ſet down; and here I have borrowed 10 (as before) which muſt be paid in the ſame manner as the other 10 was, *viz.* either by calling the 7 in the upper Rank but 6, ſaying, 5 from 6 there *Remains* 1, or elſe by ſaying 1 borrowed and 5 is 6 from 7 and there *Remains* 1, which being ſet down under its own place all is done, and the *Difference* required will be 1647=7496—5849.

Example 3.

From	830476
Take	741068
Remains	89408

By this Example you may perceive that *Cyphers* in the *Subtrahend*, viz. in the *Numbers* to be *Subſtracted*, do not Diminiſh the *Number* from whence *Subſtraction* is made. See *Page* 4.

Theſe Three Examples I preſume may be ſufficient to ſhew the young Learner the Method of *Subſtracting* whole *Numbers*; as for the Reaſon thereof it's the ſame with that of *Addition*, Page 10, *viz.* of *the Whole being Equal to all its Parts taken together.*

That is, in this *Rule* the *Number* from which *Subſtraction* is required to be made, is underſtood to be the Whole, and the *Subtrahend* or *Number* to be *Subſtracted*, is ſuppos'd to be a part of that Whole, conſequently if that Part be taken from the Whole, the *Remainder* will be the other part.

From hence is deduced the common Method of proving *Subſtraction*, by Adding together the *Subtrahend* and the *Remainder*. For if the *Sum* of thoſe Two which are here called Parts, be Equal to the *Number* from whence *Subſtraction* was made (which is here called the Whole) then the Work is Right; if not, care muſt be taken to diſcover and correct the Error.

Example.

From	59435	
Take	47608	} Add
	11827	
Proof	59435	} The *Sum* which is Equal to the *Number* from whence *Subſtraction* was made.

Or

Or from the aboveſaid Reaſon, it will be eaſy to conceive how to prove the Truth of *Subſtraction* by *Subſtraction.*

For if from	59435	being here the whole,
there be taken	47608	as part of that whole,
there will *Remain*	11827	the other part (as before)
And if from	59435	the whole, there be *Subſtracted* the
laſt part, *viz.*	11827	
there will *Remain*	47608	the firſt part, or *Number* which was required to be firſt *Subſtracted.*

From	75643		From	7000000
Take	9000		Take	986432
	66643	Remains		6013568

Sect. 4. *Of* Multiplication.

Multiplication is a *Rule* by which any given *Number* may be ſpeedily Increaſed, according to any propoſed *Number* of *Times.*

That is, One Number *is ſaid to* Multiply *another, when the* Number Multiplied *is ſo often* Added *to it ſelf, as there are* Units *in the* Number Multiplying; *and another* Number *is produced,* (Euclid. 7. Def. 15.)

To perform *Multiplication* there is required two given *Numbers,* called *Factors.*

The Firſt is that *Number* which is to be *Multiplied,* and is generally put the greateſt of the Two *Numbers,* commonly call'd the *Multiplicand.*

The other is that *Number* by which the Firſt is to be *Multiplied,* and is uſually called the *Multiplicator* or *Multiplier*; and this denotes the *Number* of *Times* that the *Multiplicand* is required to be *Added* to it ſelf. For ſo many *Units* as are contained in the *Multiplier,* ſo many times will the *Multiplicand* be really *Added* to it ſelf, (*as per* Euclid *above*). And from thence will ariſe a Third *Number,* called the *Product. But in* Geometrical Operations *its called the* Rectangle *or* Plain.

For inſtance; ſuppoſe it were required to Increaſe 6 Four times, That is, to *Multiply* 6 into or with 4, theſe Two *Numbers* are to be ſet (or placed) down as in *Addition* or *Subſtraction,*

Thus

Thus { 6 *Multiplicand*, / 4 *Multiplier*, } or *Factors*.

Product 24 *viz.* 4 times 6 is 24, as plainly appears by *Addition*, *viz.* By Setting down 6 Four times, and then adding them together into one *Sum*,

Thus

1	6
2	6
3	6
4	6

} Add

24

From hence it is Evident that Multiplication *is only a* Concise *or* Compendious *Way of* Adding *any given* Number *to it self, so often as any* Number *of Times may be proposed.*

Before any Operation can be readily performed in *Multiplication*, the several *Products* of the single *Figures* one into another must be perfectly Learn'd by Heart, *viz.* That 2 times 2 is 4, that 3 times 3 is 9, and 3 times 6 is 18, *&c.* According as they are expressed in the following *Table*; Wherein I have omitted *Multiplying* with 2, it being so very easy that any one may do it.

Multiplication Table.

3×3= 9	4×4=16	5×5=25	6×6=36	7×7=49	8×8=64
3×4=12	4×5=20	5×6=30	6×7=42	7×8=56	8×9=72
3×5=15	4×6=24	5×7=35	6×8=48	7×9=63	9×9=81
3×6=18	4×7=28	5×8=40	6×9=54		
3×7=21	4×8=32	5×9=45			
3×8=24	4×9=36				
3×9=27					

I think it needless to give any *Explanation* of this *Table*; for if the *Signs* and their *Significations* be well understood, (*vide Page* 5) it must needs be easy. Only this may be noted, that 4×3=3×4, or 7×5=5×7, *&c.*

That is, 3 times 4 is the same with 4 times 3, or 5 times 7 is the same with 7 times 5, *&c.* The like must be understood of all the rest in the *Table.*

And when all these single *Products* are so perfectly Learn'd by Heart, as to be said without pausing; you may then proceed (but not till then) to the Business of *Multiplication*; which will be found very easy, if the following *Rule* (and *Examples*) be carefully observed.

Rule.

Always begin with that Figure *which stands in the* Units *place of the* Multiplier, *and with it* Multiply *the* Figure *which stands in*

in the Units *place of the* Multiplicand; *if their* Product *be Less than Ten, ſet it down underneath its own place of* Units, *and proceed to the next* Figure *of the* Multiplicand. *But if their* Product *be above* Ten (*or* Tens) *then ſet down the Overplus only (or odd* Figure, *as in* Addition) *and bear* (*or carry*) *the ſaid* Ten *or* Tens *in mind until you have* Multiplied *the next* Figure *of the* Multiplicand, *with the ſame* Figure *of the* Multiplier; *then to their* Product *Add the* Ten *or* Tens *beared in mind, ſetting down the Overplus of their* Sum *above the* Tens, *as before: and ſo proceed on in the very ſame manner, until all the* Figures *of the* Multiplicand *are* Multiplied *with that* Figure *of the* Multiplier.

Example 1.

Suppoſe it were required to *Multiply* 3213 into or with 3.

3213 *Multiplicand*,
3 *Multiplier*, } or *Factors*.

Product 9639

Beginning at the *Units* place, ſay, 3 times 3 is 9, which becauſe it is leſs than *Ten*, ſet it down underneath its own place, and proceed to the next place of *Tens*, ſaying 3 times 1 is 3, which ſet down underneath its own place; then to the next place, *viz.* of *Hundreds*, ſaying 3 times 2 is 6, which ſet down, as before; laſtly, at the place of *Thouſands*, ſay 3 times 3 is 9, which being ſet down underneath its own place, the Operation is finiſhed; and the true *Product* is 9639=3213×3, as was required.

Example 2.

Let it be required to *Multiply* 8569 into 8. Set down theſe *Numbers* as before,

Thus { 8569
8

68552

Beginning at the *Units* place ſay, 8 times 9 is 72, ſet down the 2 underneath its own place of *Units*, and bear the 70, or 7 *Tens* in mind, and proceed to the next *Figure* of the *Multiplicand* (at which place the 7 *Tens* are only 7) ſaying 8 times 6 is 48, and the 7 carried in mind is 55; ſet down the odd 5 underneath its own place of *Tens*, and carry the 50 (which is really 500) to the next place (*viz.* of *Hundreds*) at which place it is only 5, where ſay, 8 times 5 is 40, and the 5 carried in mind is 45; ſet down the 5 underneath its own place, and carry the 40 or 4 *Tens* (which is really 4000) to the

next

ext place, *viz.* of *Thouſands*, ſaying, 8 times 8 is 64, and 4 rried in mind is 68. (Now this being the laſt place or *Figure* be *Multiplied)* ſet down the whole *Product* 68, and the Work done.

So that, 8569×8=68552, the *Product* required.

Now the Reaſon of this and all other the like Operations, ay be eaſily conceived from this which follows,

8 5 6 9 } The ſame *Factors* as before.
8 }

Here 8 times 9 is 72, as before, becauſe the 9 ſtands in the *Units* place.

Now here it is not really 8 times 6=48, but it is 8 times 60=480, becauſe the 6 ſtands in the place of *Tens*.

And here it is not 8 times 5=40, but it's really 8 times 500=4000, becauſe the 5 ſtands in the place of *Hundreds*.

Laſtly, becauſe the 8 in the *Multiplicand* ſtands in the place of *Thouſands*, it's therefore 8 times 8000=64000, and not 8 times 8=64.

6 8 5 5 2 } The *Sum* of the particular *Products*, which gives the true *Product*, as before.

By what hath been already ſaid, with a little Conſideration ad to the *Examples*, I preſume the Learner may eaſily under- tand how to *Multiply* whole *Numbers* with any ſingle *Figure*. And when it is requir'd to *Multiply* with more than one; Then o many Figures as there are in the *Multiplier*, ſo many parti- ular *Products* there muſt be.

That is, all the *Figures* of the *Multiplicand* muſt be *Multi- lied* with every ſingle *Figure* of the *Multiplier* as if there ere but one ſingle *Figure*; and the *Sum* of all thoſe particular *Products*, will be the True *Product* required; but in thoſe perations, great Care muſt be taken in ſetting down the par- icular *Products* (which ariſe by each *Multiplying Figure)* in heir proper places. Which will be eaſily done, if the follow- ng Directions be carefully obſerved.

Viz. Always place the firſt *Figure* (or *Cypher*) of every particular *Product*, directly underneath the *Multiplying Figure*. Or thus:

The Firſt Figure (*or* Cypher) *of the ſecond particular* Product *muſt ſtand directly under the ſecond* Figure *(or place) of the Firſt* Product; *and the Firſt* Figure (*or* Cypher) *of the Third*

particular Product, *muſt ſtand directly underneath the Third* Figure *of the Firſt* Product: *And ſo on until all is done.*

Now the Reaſon of placing the firſt *Figure* of every particular *Product* in this order, will be very obvious to any one that conſiders the laſt *Example*; wherein the *Cyphers* are only ſet down to ſhew the true diſtance of the firſt *Figure* in each particular *Product* from the *Units* place. And altho' it is not uſual to ſet down *Cyphers* in this manner; yet they are always ſuppos'd to be there: That is, their Places are always left void, as in the two following *Examples*; wherein I have placed *Points* inſtead of *Cyphers*.

Example 3.

Let it be required to *Multiply* 78094, into or with 7563.

78094 } *Factors.*
7563 }

234282	The Firſt particular *Product* with	3
468564·	The Second particular *Product* with	60
390470··	The Third particular *Product* with	500
546658···	The Fourth particular *Product* with	7000
590624922	The *Total*, or true *Product* required.	

Example 4.

Suppoſe it be required to *Multiply* 57498 into 60008.

57498
60008

459984	The *Product* with	8
344988····	The *Product* with	60000
3450339984	=57498×60008 as was required.	

Here you may obſerve, that I paſs over the *Cyphers*, and only take care of placing the firſt *Product* of the laſt *Figure*, viz. of 60000 according to the foregoing Directions.

When there is a *Cypher* or *Cyphers*, to the Right-hand either of the *Multiplicand*, or *Multiplicator*, or to both; in that caſe *Multiply* the *Figures* as before; neglecting the *Cyphers* until the particular *Products* are added together; Then to their *Sum* annex ſo many *Cyphers* as are in either or both the *Factors*. As in theſe:

Exam-

Example 5.	*Example* 6.	*Example* 7.
9538	87600	785000
4600	79	56900
57228	7884	7065
38152	6132	4710
		3925
43874800	6920400	44666500000

Take a few Examples without their Work at large.

75649×579=43800771
687000×356=244572000
530674×45007=23884044718
7901375×30000=237041250000
537084000×590700=317255518800000
102030405×504030201=5142640554026140 5
987654321×123456789=121932641112635269

Note, If it be required to *Multiply* any *Number* with 10, 100, 1000, 10000, &c. it's only annexing the *Cyphers* of the *Multiplier* to the *Figures* of the *Multiplicand*, and the Work is done.

Thus { 578×10 =5780 . 578×1000 =578000
{ 578×100=57800. 578×10000=5780000, &c.

These *Examples* (being well understood) are sufficient to instruct the Learner, in all the Varieties that can happen in *Multiplying* of whole *Numbers*, according to the Method generally practised: However it may not be amiss to shew here how *Multiplication* may be performed (with many *Figures*) by *Addition* only.

Example.

Let it be required to *Multiply* 879654 into 79863.

In order to perform this (or any other Operation of this kind) by *Addition* only; you must make a Tariffa or small Table of the given *Multiplicand*, in this manner:

First, Make a small Column, and in it place gradually downward the Nine single *Figures*; viz. 1, 2, 3, 4, 5, &c.

Then againſt the *Figure* 1, ſet down the *Multiplicand* (which in this *Example* is 879654) and againſt the *Figure* 2, ſet down the double of the *Multiplicand*, found by *Adding* it to its ſelf; To this double *Add* the *Multiplicand*, ſetting down their *Sum* againſt the *Figure* 3. And ſo proceed on by a continued *Addition* until there be Ten times the *Multiplicand* in the *Table*, which if the Work is true, will be the *Multiplicand* it ſelf with a *Cypher* to the Right-hand of it (as in the Annexed Table) this being done, it will be eaſie to conceive, that the *Figures* in the ſmall Column of the *Table*, do reſpectively repreſent thoſe of the *Multiplier*; And that the *Numbers* againſt any of thoſe *Figures* in the ſmall Column, will be the true *Product* of the *Multiplicand* agreeing to any *Figure* of the *Multiplier*; as plainly appears by the Work of this *Example*,

1	879654
2	1759308
3	2638962
4	3518616
5	4398270
6	5277924
7	6157578
8	7037232
9	7916886
10	8796540

Then 879654 } The *Factors* as before.
79863 }

Againſt 3, in the *Table* is	2638962	=879654×3
Againſt 6, is	5277924	=879654×60
Againſt 8, is	7037232	=879654×800
Againſt 9, is	7916886	=879654×9000
Againſt 7, is	6157578	=879654×70000
The Product required	70251807402	=879654×79863

Note, This Method of Tabulating the *Multiplicand*, is both eaſie and certain; being neither ſubject to Errors, nor burthenſome to the Memory, and therefore in large Calculations it may be found very uſeful. But for common Practice the uſeful Method (as in *Page* 18, *&c.*) is beſt, and to be preferr'd before this.

Moſt *Maſters* that teach (and ſeveral *Authors* that write of) *Arithmetick*, do teach to prove the Truth of *Multiplication*, by caſting away all the *Nines* that are contain'd in both the *Factors*, and their *Product*; but becauſe that Method is very erroneous, as might be eaſily ſhew'd; I ſhall therefore omit inſerting it, and leave the Proof of *Multiplication* to the next *Section*, wherein (I preſume) the Reaſon and Proof, both of it, and *Diviſion*, will plainly appear,

Sect.

Sect. 5. *Of* Division.

Division is a *Rule* by which one *Number* may be ſpeedily *Subſtracted* from another, ſo many times as it is contained therein.

That is, It ſpeedily diſcovers how often one *Number* is contained (or may be found) in another: And to perform that there are required Two Numbers to be given.

1. The one of them is that *Number* which is propoſed to be *Divided*, and is called the *Dividend*.

2. The other is that *Number* by which the ſaid *Dividend* is to be *Divided*, and is called the *Diviſor*.

And by comparing theſe Two, *viz.* the *Dividend* and the *Diviſor* together, there will ariſe a Third *Number*, called the *Quotient*; which ſhews how often the *Diviſor* is contained in the *Dividend*, or into what *Number* of Equal Parts the *Dividend* is then divided. Therefore,

Diviſion *is by* Euclid *fitly term'd the Meaſuring of one* Number *by another*, *viz.* one *Number* is ſaid to Meaſure another by that *Number*, which when it *Multiplies*, or is *Multiplied* by it, it produceth. *Euclid.* 7. *Def.* 23.

And, if a Number *meaſuring another*, Multiply, *that* Number *by which it Meaſureth, or be* Multiplied *by it, it produceth the* Number *which it meaſureth.* Euclid. 7. Axiom 9.

That is to ſay, If that *Number* which *Divides* another, (called the *Diviſor*) be *Multiplied* with the *Number* which is produced by *Diviſion* (called the *Quotient*) their *Product* will be the *Number Divided* or *Dividend*. Whence it follows, that *Diviſion* and *Multiplication* are the Converſe and Direct Contrary one to another (as *Subſtraction* is to *Addition*) and do mutually prove the Truth of each others *Operations*.

I ſhall therefore make choice of the foregoing *Examples* in *Multiplication*, in order (as I preſume) to render the Buſineſs of *Diviſion* more plain and eaſie.

Firſt, Let it be required to find how often 6 is contained in 24. That is, to *Divide* 24 by 6.

N. B. Always place down the given *Numbers* in this order; Firſt ſet down the *Diviſor*, and to the Right-hand of it draw a crooked Line; then ſet down the *Dividend*, and to the Right of it draw another crooked Line, in which muſt be placed the *Quotient Figure*, or *Figures* as they become found.

Dividend

Dividend

Thus, *Diviſor* 6) 24 (4 the *Quotient*.

Here I conſider how many times 6 there is in 24, and find it 4, *viz.* 4 times 6 is 24, therefore 4 is the true *Quotient* or *Anſwer* required.

This is apparent by *Subſtraction*, as in the Margin, where 24 the *Dividend* being ſet down, and from it 6, the *Diviſor* is continually *Subſtracted* ſo often as it can be, which is juſt 4 times. Therefore 4 is the true *Quotient* or *Anſwer* required.

Compare this with the *Example*, *Page* 15.

	24
1	6
	18
2	6
	12
3	6
	6
4	6
	0

Corollary.

From hence it is evident; that *Diviſion* is but a *Conciſe* or *Compendious Method* of *Subſtracting* one *Number* from another, ſo often as it can be found therein; for if the *Diviſor* be continually *Subſtracted* from the *Dividend*, accounting an *Unit* (or 1) for each time it is *Subſtracted* (as above) the *Sum* of thoſe *Units* will be the *Quotient*.

All Operations in *Diviſion* do begin contrary to thoſe of *Multiplication*, *viz.* at the Firſt *Figure* to the Left-hand, or that of the higheſt Value, and Decreaſe the *Dividend* by a repeated *Subſtraction* of each *Product* ariſing from the *Diviſor* when *Multiplied* into the *Quotient Figure*. And the only difficulty in *Diviſion* of whole *Numbers* (or indeed of any *Numbers*) lies in making choice of ſuch a *Quotient Figure*, as is neither Too big, nor Too little; and that may be eaſily obtained by obſerving the following *Rule*, which hath two *Caſes*.

Rule.

Caſe 1. *As often as the Firſt* Figure *of the* Diviſor *is taken from the Firſt* Figure *of the* Dividend: *So often muſt the Second* Figure *of the* Diviſor *be taken from the Second* Figure *of the* Dividend, *when it's joyned with what* Remains *of the Firſt. And as often muſt the Third* Figure *of the* Diviſor *be taken from the Third* Figure *of the* Dividend, *&c.*

But if the Firſt *Figure* of the *Diviſor* cannot be taken from the Firſt *Figure* of the *Dividend*, Then,

Caſe

Case 2. *So often as the First* Figure *of the* Divisor, *is taken from the Two First* Figures *of the* Dividend, *so often must the Second* Figure *of the* Divisor *be taken from the Third* Figure *of the* Dividend, *when it's joyned with what* Remain'd *of the Second: And so often must the Third* Figure *of the* Divisor *be taken from the Fourth* Figure *of the* Dividend, *&c.*

That is, the *Quotient Figure* must be such, as being *Multiplied* into the *Divisor*, will *Produce* a *Product* Equal to such a part of the *Dividend* as is then taken for that *Operation:* But if such a *Product* cannot be exactly found, then the next less must be taken, and ordered, as in the following *Examples*: of which let that in *Page* 16 be the first, wherein there was given 8569 the *Multiplicand*, and 8 the *Multiplier*. To find the *Product* 68552. Let us here suppose the said *Product* 68552, and 8 the *Multiplier*, both given; thence to find the *Multiplicand*. That is, Let it be required to *Divide* 68552 by 8.

Dividend

Divisor 8) 68552 (*Quotient* when found.

According to the *Rule*, *Case* 1. I compare 8 the *Divisor* with 6 the First *Figure* of the *Dividend*, and finding I cannot take it from that; I then consider (by *Case* 2.) how often 8 can be taken from 68, the Two first *Figures* of the *Dividend*, and find it may be taken 8 times; for 8 times 8 is 64, being the greatest *Product* of 8 (into any *Figure*) that can be taken from 68. I therefore place 8 in the *Quotient*, and with it *Multiply* 8 the *Divisor*, setting down their *Product* underneath the said Two First *Figures* of the *Dividend*, *Substracting* it from them, and then the Work will stand

Thus 8) 68552 (8
64

4

In order to a Second *Operation*, I make a *Point* under the next *Figure* of the *Dividend*, *viz.* under the 5, and bring it down underneath its own place to the *Remainder* 4, which will by that means become 45. Then I consider how many times 8 can be taken from 45, and find it may be 5 times; for 5 times 8 is 40, I therefore place 5 in the *Quotient*, and with it *Multiply* 8 the *Divisor*, setting down and *Substracting* their *Product*, as before. Then the Work will stand

Thus

Thus 8) 68552 (85
64
45
40
5

For a Third *Operation*, I make a *Point* under the next *Figure* of the *Dividend*, *viz.* under the 5, and bring it down, as before, proceeding in all reſpects, as before; and then the Work will ſtand

Thus 8) 68552 (856
64
45
40
55
48
7

Laſtly, I point and bring down the 2, *viz.* the laſt *Figure* of the *Dividend* to the *Remainder* 7, which will then become 72, and proceeding as in the other *Operations*, I find that 8 the *Diviſor* can be taken juſt nine times from 72, and the Work is finiſhed, and will ſtand

Thus 8) 68552 (8569
64
45
40
55
48
72
72
(0)

The True *Quotient* is found to be 8569, being exactly the *Eighth* part of 68552, or the *Multiplicand* of the propoſed *Example* of *Multiplication*. As was required.

The Reaſon of the *Operations* will be very plain to any one that will a little conſider of it as follows.

Diviſor

Diviſor 8) 6 8 5 5 2 8000. The Firſt *Quotient* or *Figure.*

Subſtract 6 4 0 0 0 } This *Product* of the *Diviſor* into the *Quotient* is 64000. *viz.* 8 times 8000. the *Quotient Figure* being always of the ſame *Value* or *Degree* with that *Figure* under which the *Unit*'s place of its *Product* ſtands

Diviſor 8) 4 5 5 2 (500. The Second *Quotient Figure.*
Subſtract 4 0 0 0 { And here the *Product* is 4000. *viz.* 8 times 500. not 8 times 5, *&c.*

Diviſor 8) 5 5 2 (60. The Third *Quotient Figure.*
Subſtract 4 8 0 { Alſo here the *Product* is 480. *viz.* 8 times 60. for the *Reaſons* aboveſaid.

Diviſor 8) 7 2 (9. The Fourth *Quotient Figure.*
Subſtract 7 2 { Now here the *Product* is but 72. *viz.* 9 times 8. becauſe the 9 ſtands in the place of *Units.*

Remains (0 0) Now the *Sum* of all the ſeveral *Quotients*, *viz.* 8000+500+60+9=8569. as before.

If the *Proceſs* of this *Example* be well conſidered and compared with that of *Multiplication*, *Page* 17. it will evidently appear to be only the *Converſe* of that; for the particular *Products* are alike in both, only that which is *Laſt* there, is *Firſt* here; there they are *Added*, here they are *Subſtracted.* So that whoever underſtands the true *Reaſon* of the one, muſt needs underſtand the *Reaſon* of the other, and then *Diviſion* will become very *Eaſy*, although the *Diviſor* conſiſt of ſeveral places of *Figures.*

Example.

Let it be required to *Divide* 590624922 by 7563.

Dividend

Diviſor 7563) 590624922 (

'Tis plain at ſight, that 7563 the *Diviſor*, cannot be taken from 5906. the like *Number* of *Figures* in the *Dividend.*

Therefore, by the *Second Caſe* of the *Rule* (*Page* 23.) there muſt be allowed Five *Figures* of the *Dividend*, *viz.* 59062 for the *Firſt Operation* or *Quotient*; that ſo the *Firſt Figure* 7 of the *Diviſor* may be taken out of the Two *Firſt Figures*, *viz.* 59 of the *Dividend*, &c.

Then I proceed (*per Caſe* 2.) and conſider how often 7 may be taken from 59. and find it may be taken 8 times, for 8 times 7 is but 56. which I mentally *Subſtract* from 59. and there *Remains* 3; to this 3 I mentally adjoyn the *Third Figure* of the *Dividend*, *viz.* 0. which makes it 30. out of which I muſt take the *Second Figure* of the *Diviſor*, *viz.* 5. ſo often as I took the 7 from 59. which was 8 times. But that cannot be, for 8 times 5 is 40. which is more than 30. therefore 8 is too big a *Figure* to be placed in the *Quotient*; Yet, hence I conclude, that the next Leſs, *viz.* 7 may be taken without any further *Tryal*. I therefore place 7 in the *Quotient*, and with it *Multiply* the *Diviſor*, ſetting down their *Product* under the *Dividend*, and *Subſtract* it from thence, as in the other *Example*, and then the *Work* will ſtand

Thus 7563) 590624922 (7
52941

6121

In order to a *Second Operation*, I make a *Point* under the next *Figure* of the *Dividend*, *viz.* under the 4. and bring it down to the *Remainder* 6121. which will then become 61214. with which I proceed in all reſpects as I did before with the 59062. and find the next *Quotient Figure* will be 8. with which I *Multiply* the *Diviſor*, &c. and *Subſtract* their *Product* from the ſaid 61214. Then the *Work* will ſtand

Thus 7563) 590624922 (78
52941

61214
60504

710

To this *Remainder* 710. I point and bring down the next *Figure* of the *Dividend*, *viz.* 9. which makes it 7109; now becauſe the *Diviſor* 7563 cannot be taken from 7109. I therefore place a *Cypher* in the *Quotient*.

And this muſt always be carefully obſerved, viz. *That for every* Figure *or* Cypher, *which is brought down from the* Dividend, *in order to a new* Operation, *there muſt always be either a* Figure *or* Cypher, *ſet down in the* Quotient. Then the Work will ſtand

Thus

Thus 7563) 590624922 (780
52941··
61214
60504
7109

To this 7109, I bring down another *Figure* of the *Dividend*, viz. 2, and then it will become 71092; then I consider how often 7 can be taken from 71, &c. (just as at the first Operation) and find it may be taken 9 times, therefore I set down 9 in the *Quotient*, and with it *Multiply* the *Divisor*, setting down and *Subtracting* their *Product*, as before; Then the Work will stand

Thus 7563) 590624922 (7809
52941···
61214
60504
71092
60067
3025

To this *Remainder* 3025, I point and bring down the last *Figure* 2 of the *Dividend*, which makes it 30252; then proceeding in all respects as before, I find the *Quotient Figure* to be 4, with it I *Multiply* the *Divisor*, setting down and *Subtracting* their *Product* as before, and then the Work will stand

Thus 7563) 590624922 (78094
52941····
61214
60504
71092
68067
30252
30252
(00000)

Here the Work is ended, and I find the *Quotient* to be 78094, being the true *Multiplicand* of the propos'd *Example* of *Multiplication*, Page 18.

That is, 7563 is contained in 590624922, just 78094 times, &c.

If the Work of this *Example* be conſidered and compared with the *Rule* (Page 22.) the whole Buſineſs of *Diviſion* will be eaſy; for indeed the only Difficulty (as I ſaid before) lies in making choice of a True *Quotient Figure*, which cannot well be done according to the Common Method of *Diviſion*, without Trials, yet thoſe Trials need not be made with the whole *Diviſor* (as appears by this laſt *Example*) for by the Two Firſt *Figures* of the *Diviſor* all the reſt are generally regulated; except the Second *Figure* chance to be 2, 3, or 4, and at the ſame time the Third *Figure* be 7, 8, or 9, then indeed reſpect muſt be had to the Third *Figure*, according as the *Rule* directs.

However, if thoſe Trials are thought too troubleſome, they may be avoided, and the ſame *Quotient Figures* may both eaſily and certainly be found by help of ſuch a ſmall *Table* made of the *Diviſor*, as was of the *Multiplicand* in *Page* 20.

Example 4.

Let it be required to *Divide* 70251807402 by 79863. *See the Example of Multiplication, page* 20, *and as there directed make a Table of the Diviſor* 79863, Thus,

	Diviſor.	*Dividend.*	
1	79863)	70251807402	(879654. *Quotient.*
2	159726	638904·····	
3	239589	636140	
4	319452	559041	
5	399315	770997	
6	479178	718767	
7	559041	522304	
8	638904	479178	
9	718767	431260	
10	798630	399315	
		319452	
		319452	
		(000000)	

The Work of this Operation I preſume may be eaſily underſtood. For thoſe *Figures* in the *Table* are the *Products* of the *Diviſor* into all the 9 *Figures*; conſequently thoſe *Figures* in the ſmall Column do ſhew what *Figure* is to be placed in the *Quotient*; without any doubtful Trials of the *Diviſor* with the *Dividend*, as before.

This Method of Tabulating the *Diviſor* may be of good Uſe to a Learner. Eſpecially until he is well practiſed in *Diviſion*; yea, and even then if the *Diviſor* be large, and a *Quotient* of many *Figures* be required; as in Reſolving of high *Æquations*, and Calculating of *Aſtronomical Tables*, or thoſe of Intereſt, &c.

Hitherto

Hitherto I have made choice of *Examples* wherein the *Dividend* is truly Measured or *Divided* off, by the *Divisor*, without leaving any *Remainder*, being those as were composed of the *Divisor* and *Quotient*. But it most usually falls out, that the *Divisor* will not exactly measure the *Dividend*; in that case the *Remainder* (after *Division* is ended) must be set over the *Divisor* with a small Line betwixt them adjoyning to the *Quotient*.

Example 5.

Suppose it were required to *Divide* 379 by 5.

5) 379 (75$\frac{4}{5}$ the Remainder. the Divisor.
35·

29
25

Remains (4)

Example 6.

Again, Let it be required to *Divide* 43789 by 67.

67) 43789 (653$\frac{38}{67}$ the True *Quotient* required
402··

358
335

239
201

Remains (38)

How such *Remainders* thus placed over their *Divisors* (which are indeed *Vulgar Fractions*) may be otherwise managed, shall be shew'd farther on.

N. B. When the *Divisor* happens to be an *Unit*, viz. 1 with *Cypher* or *Cyphers* annexed to it, As 10, 100, 1000, &c. *Division* is truly performed by cutting off with a Point or Comma, so many *Figures* of the *Dividend* as there are *Cyphers* in the *Divisor*; then are those *Figures* so cut off to be accounted a *Remainder*, and the rest of the *Figures* in the *Dividend* will be the true Quotient required, because an *Unit* or 1 doth neither *Multiply* nor *Divide*.

Example 7.

Let it be required to *Divide* 57842 by 100. The Work may stand thus, 100) 578,42 the *Quotient* Required, or thus 100) 57842 (578$\frac{42}{100}$ the same as before.

Hence it follows, that if any *Divisor* have *Cyphers* to the Right-hand of it, you may cut off so many of the last *Figures* in

in the *Dividend*, and *Divide* the other *Figures* of the *Dividend* by thoſe *Figures* of the *Diviſor* that are left when the *Cyphers* are omitted. But when *Diviſion* is ended, thoſe *Cyphers* omitted in the *Diviſor*, and the *Figures* cut off in the *Dividend* are both to be reſtored to their own places.

Example 8.

Suppoſe it were required to *Divide* 675469 by 5400.

5400) 675469 (125
54··
135
108
274
270
Remains (4)

But the True *Remainder* is 469. Conſequently the True *Quotient* is $125\frac{469}{5400}$.

As to the manner of proving the Truth of any Operation, either in *Multiplication* or *Diviſion*, I preſume it may be eaſily underſtood, by what is deliver'd in *Page* 21, compared with the Three Firſt *Examples* of *Diviſion*; For from thence it will be eaſy to conceive, that if the *Diviſor* and *Quotient* be *Multiplied* together, their *Product* (with what *Remains* after *Diviſion* being added to that *Product*) will be equal to the *Dividend*. As in the Fifth *Example*, wherein the *Dividend* is 379, the *Diviſor* is 5, the *Quotient* is 75, and the *Remainder* is 4.

I ſay, $75\times5=375$, to which *Add* the *Remainder* 4, it will be 379.

Again, in the Sixth *Example*, the *Diviſor* is 67, the *Quotient* is 653, and the *Remainder* is 38.

Then $653\times67=43751$, and $43751+38=43789$ the *Dividend*, &c.

There are ſeveral uſeful Contractions, both in Diviſion *and* Multiplication, *which I have purpoſely omitted until I come to treat of* Decimal Arithmetick. Alſo I have omitted the buſineſs of Evolution or Extracting of *Roots*, until further on; and ſo ſhall conclude this *Chapter* with a few *Examples* of *Diviſion* unwrought at large, leaving them for the Learners Practice.

579) 43800771 (75649.
Or 75649) 43800771 (579

45007)

45007) 23884044718 (530674.
Or 530674) 23884044718 (45007.
356) 244572000 (687000.
59600) 57659066400 (967434.
10000) 6795438200000 67954382.
79) 282016 (3569$\frac{65}{79}$.

CHAP. III.

Concerning **Addition** *and* **Subtraction** *of* Numbers *of different* Denominations, *and how to* Reduce *them from one* Denomination *to another.*

SECT. 1.

1. *Of* Engliſh *Coin.*

THE leaſt Piece of *Money* uſed in *England* is a *Farthing*, and from thence ariſeth the reſt, as in this *Table.*

Farth.
4 = 1 *d. Penn.*
48 = 12 = 1 *s. Shill.*
960 = 240 = 20 = 1 *l. Pound Sterling.*

And { 5 *s.* is a *Crown.* / 10 *s.* is an *Angel.* / 6 *s.* 8 *d.* a *Noble.* / 13 *s.* 4 *d.* a *Mark.*

Note, When *l. s. d. q.* are placed over *(or to the Right-hand of) Numbers*, they denote thoſe *Numbers* to ſignify *Pounds*, *Shillings*, *Pence*, and *Farthings.*

l. s. d. q.
As 35 10 6 2. Or 35 *l.* 10 *s.* 6 $\frac{1}{2}$ *d.* Either of theſe do ſignify 35 *pounds*, 10 *ſhillings*, 6 *pence*, 2 *farthings.*

The ſame muſt be underſtood of all the following *Characters*, belonging to their reſpective *Tables*, *viz.* Of *Weights*, *Measures*, &c.

2. *Troy Weight.*

The Original of all *Weights* uſed in *England*, was a *Corn* of *Wheat* gathered out of the *Middle* of the *Ear*, and being well dried, 32 of them were to make one *Penny Weight*, 20 *Penny Weight*

Weight one *Ounce*, and 12 *Ounces* one *Pound Troy*. *Vide Statutes* of 51 *Hen.* 3. 31 *Edw.* 1. 12 *Hen.* 7.

But in latter Times it was thought ſufficient to *Divide* the aforeſaid *Penny Weight* into 24 Equal *Parts*, called *Grains*, being the leaſt *Weight* now in common Uſe; and from thence the reſt are computed as in this *Table*.

Gr. Gra.				
24 =	1 P.W. *Penny Weight.*			
480 =	20 =	1 Oz. *Ounce.*		
7560 =	240 =	12 =	1 lb *Pound.*	

Note, By *Troy Weight* are weighed *Jewels*, *Gold*, *Silver*, *Corn*, *Bread*, and all *Liquors*.

Beſides the common *Diviſions* of *Troy Weight*, I find in *Angliæ Notitia*, or, *The Preſent State of England*, Printed in the Year 1699, that the *Moneyers* (*as that Author calls them*) do *Subdivide* the *Grain*

Thus
24 *Blanks* = 1 *Periot.*
20 *Periots* = 1 *Droite.*
24 *Droites* = 1 *Mite.*
20 *Mites* = 1 *Grain*, &c. as before.

3. *Apothecaries Weights.*

The *Apothecaries Divide* a *Pound Troy*, as in this *Table*:

Gr. Grain.				
20 =	1 ℈ *Scruple*			
60 =	3 =	1 ʒ *Dram*		
480 =	24 =	8 =	1 ℥ *Ounce*	
5760 =	288 =	96 =	12 =	1 lb *Troy*, the ſame as before.

By theſe *Weights* the *Apothecaries* Compound their Medicines, but Buy and Sell their Drugs by *Averdupois Weight*.

4. *Averdupois Weight.*

When *Averdupois Weight* became firſt in *Uſe*, or by what *Law* it was firſt ſettled, I cannot find out in the *Statute Books*; but on the contrary I find that there ſhould be but one *Weight* (*and one Meaſure*) uſed throughout this *Realm*, *viz.* that of *Troy*, (*Vide* 14 *Ed.* 3. and 17 *Ed.* 3.) So that it ſeems (*to me*) to be firſt introduced by *Chance*, and ſettled by *Cuſtom*, *viz.* from giving good or large *Weight* to thoſe *Commodities* as are uſually weighed by it, which are ſuch as are either very *Coarſe* and *Droſſy*, or very

ry subject to waste: As, all kind of Grocery Wares. And itch, *Tar*, *Rosin*, *Wax*, *Tallow*, *Flax*, *Hemp*, &c. *Copper*, *Tin*, eel, *Iron*, *Lead*, &c. Also *Flesh*, *Butter*, *Cheese*, *Salt*, &c. To ese and the like (I presume) it was thought convenient to low a greater *Weight* than what the Laws had provided, which ppen'd to be about a Sixth part more: For I found by a very ce Experiment, that one *Pound Averdupois* is Equal to 14 unces, 11 *Penny Weight*, and 15½ *Grains Troy*. And it is w computed as in the following Table.

Drams					
16 = 1	*Ounces*				
256 =	16 = 1	lb	*Pounds.*		
28672 =	1792 =	112 = 1	*C. Hundred.*		
573440 =	35840 =	2240 =	20 = 1	*Tun.*	

And lb: 14 = *a Stone*; 28 = ¼ *of C.*; 56 = ½ *of C.*; 84 = ¾ *of C.*

5. *Long Measure.*

As the least Part of *Weight* came at first from a *Wheat Corn*, so (it's generally said) the least Part of *Long Measure* was at First a *Barly Corn*, taken out of the Middle of the Ear, and being well dried Three of them in length were to make one *Inch*; and thence the rest, as in this Table.

Barly Corns						
3 = 1	*In. Inches*					
36 =	12 = 1	*F. Feet*				
108 =	36 =	3 = 1	*Y. Yards*			
594 =	198 =	16½ =	5½ =	*P. Poles*		
23760 =	7920 =	660 =	220 =	40 = 1	*Furlong*	
190080 =	63360 =	5280 =	1760 =	320 =	8 = 1	*Mile.*

And: 4 *Nails* = ¼ of a *Yard*; 1¼ *Yard* = 1 *Ell*; 2 *Yards* = 1 *Fathom*.

Note, That Forty *Poles* (or *Perches*) in Length, and Four in eadth do make a Statute *Acre* of *Land*.

That is, 220 *Yards*, *Multiplied* into 22 *Yards* = 4840 Square *rds* are a Statute *Acre*.

And according to the Transactions of the *French Academy*, *nno* 1687, a *Paris Foot Royal* is = 12,8 *Inches English*; Six those Feet make a *Toise*; and 57060 *Toise* = 365184 *English eet*, are the *Measure* of one Degree of a great Circle upon the rface of the Earth. So that one Degree is 69 *Miles* and 288 *ards*, which is very near to our Country-man Mr. *Norwood*'s xperiment made betwixt *London* and *York*, *Anno* 1635; who ound that 367196 *Feet* = 69 *Miles*, and 958 *Yards* do make a

Degree. And not 60 *Miles*, according to the common received Opinion and Practice of the *Navigators* and *Seamen*.

Hence, according to the *French* Account, the Circumference of the Earth (ſuppoſing it to be a true *Spherical Figure*) is 24899 *Engliſh Miles*.

6. *Of Liquid Meaſures.*

All *Meaſures* of Capacity, both Liquid and Dry, were at firſt made from *Troy Weight*, *Vide Statutes* 9 *H.* 3. 51 *H.* 3. 12 *H.* 7. *&c.* wherein it is Enacted, that Eight *Pound Troy Weight* of *Wheat*, gathered out of the Middle of the Ear, and well Dried, ſhould make one *Gallon* of *Wine Meaſure*: And that there ſhould be but one *Meaſure* for *Wine*, *Ale*, and *Corn*, throughout this Realm (*Vide Stat.* 14 *Ed.* 3. 15 *Ric.* 2.) But Time and Cuſtom hath alter'd *Meaſures*, as they have done *Weights* (and perhaps for one and the ſame Reaſon) for now we have Three different *Meaſures*, viz. one for *Wine*, one for *Ale* or *Beer*, and one for *Corn*.

I have inſerted *Tables* of each as they are now computed by *Cubick Inches*, and practiſed in the Art of *Gauging*, &c.

The common *Wine Gallon* ſealed at *Guild-Hall* in *London*; by which all *Wines*, *Brandies*, *Spirits*, *Strong-waters*, *Mead*, *Perry*, *Cyder*, *Vinegar*, *Oyl*, and *Honey*, &c. are Meaſured and Sold; is ſuppoſed to contain 231 *Cubick Inches*, and from thence the reſt are computed, as in this Table.

Cubick Inches							
231	= 1	*G. Gallons*					
9702	= 42	= 1	*Tierce*				
14553	= 63	= 1½	= 1	*Hogſhead*			
19404	= 84	= 2	= 1⅓	= 1	*Puncion*		
29106	= 126	= 3	= 2	= 1½	= 1	*Butt or Pipe*	
58212	= 252	= 6	= 4	= 3	= 2	= 1	*Tun.*

Gallons.

Note, { 18 = 1 *Rundlet*, and 31½ makes a *Wine* or *Vinegar Barrel.* (*Vide* 1 *R.* 3.)

But Dr. *Wybard* in his Tectometry, *Page* 289, doth ſuppoſe the *Wine Gallon* to contain but 224, or 225 *Cubick Inches* at the moſt, and purſuant to this Account an Experiment was made by Mr. *Richard Walker* and Mr. *Philip Shales*, two General Officers in the Exciſe. They cauſed a Veſſel to be very exactly made of Braſs, in Form of a *Parallelopipedon*, each ſide of its Baſe was 4 *Inches*, and its Depth 14 *Inches*; ſo that its juſt Content was 224 *Cubick Inches*. This Veſſel was produced at *Guild-Hall* in *London* (*May* 25. 1688.) before the *Lord-Mayor*, the *Commiſſioners* of *Exciſe*, the Reverend Mr. *Flamſtead* Aſtr. Reg.

Mr.

Mr. *Halley*, and several other ingenious Gentlemen, in whose Presence Mr. *Shales* did exactly fill the foresaid Brazen Vessel with clear Water, and very carefully emptied it into the old Standard *Wine Gallon* kept in *Guild-Hall*, which did so exactly fill it, that all then present were fully satisfied the *Wine Gallon* doth contain but 224 *Cubick Inches*. (*This notable Experiment I saw tried.*) However, for several Reasons, it was at that time thought convenient to continue the former supposed Content of 231 *Cubick Inches* to be the *Wine Gallon*, and that all Computations in *Gauging* should be made from thence, as above.

The *Beer* or *Ale Gallon* (which are both one) is much larger than the *Wine Gallon*, it being (as I presume) made at first to correspond with *Averdupois Weight*, as the *Wine Gallon* did with *Troy Weight*: For (as I said before *Page* 33.) one *Pound Averdupois* is Equal to 14 *Ounces* 12 *Penny Weight Troy*, very near.

And, as one *Pound Troy* is in proportion to the *Cubick Inches* in a *Wine Gallon*, so is one *Pound Averdupois* to the *Cubick Inches* in an *Ale Gallon*. That is, $12 : 231 :: 14\frac{12}{20} : 281\frac{1}{2}$, very near the *Cubick Inches* contained in an *Ale Gallon*, as appears from an Experiment made by one *Nicholas Gunton*, General *Gauger* in the *Excise*, about 44 Years ago, who by such a Vessel mentioned before in the last *Page*, did find the Standard *Ale-Quart* (kept in the *Exchequer*, Vid. 12 Car. 2.) to contain just $70\frac{1}{2}$ *Cubick Inches*, consequently the *Ale-Gallon* must contain 282 *Cubick Inches*, and from thence the following *Tables* are computed.

Ale-Measure.

Cubick Inches					
282	= 1				*Gallon*
2256	= 8	= 1			*Firkin*
4512	= 16	= 2	= 1		*Kilderkin*
9024	= 32	= 4	= 2	= 1	*Barrel*
13536	= 48	= 6	= 3	= 1½ = 1	*Hogshead.*

Note, *A Firkin of Soap and of Herrings are the same with that of Ale.*

Beer-Measure.

Cub. Inches					
282	= 1				*Gallon*
2538	= 9	= 1			*Firkin*
5076	= 18	= 2	= 1		*Kilderkin*
10152	= 36	= 4	= 2	= 1	*Barrel*
15228	= 54	= 6	= 3	= 1½ = 1	*Hogshead.*

N. B. This Diſtinction or Difference betwixt *Ale* and *Beer Meaſure*, is now only uſed in *London*. But in all other Places of *England* the following *Table* of *Beer* or *Ale*, whether it be ſtrong or ſmall, is to be obſerved, according to a Statute of *Exciſe* made in the Year 1689.

Cub. Inches		
282	= 1	*Gallon*
2397	= 8½ = 1	*Firkin*
4794	= 17 = 2 = 1	*Kilderkin*
9588	= 34 = 4 = 2 = 1	*Barrel*
14382	= 51 = 6 = 3 = 1½ = 1	*Hogſhead.*

7. Of *Dry Meaſure.*

Dry Meaſure is different both from *Wine* and *Ale Meaſure*, being as it were a Mean betwixt both, tho' not exactly ſo; which upon Examination I find to be in proportion to the aforeſaid old Standard *Wine Gallon*, as *Averdupois Weight* is to *Troy Weight*; That is, as one *Pound Troy* is to one *Pound Averdupois*, ſo is the *Cubick Inches* contained in the old *Wine Gallon*: To the *Cubick Inches* contained in the *Dry* or *Corn Gallon*.

Viz. 12 : $14\frac{12}{20}$: : 224 : $272\frac{1}{5}$, which is very near to $272\frac{1}{2}$ the common received Content of a *Corn Gallon*, altho' now it is otherwiſe ſettled by an Act of Parliament made in *April* 1697 the Words of that Act are theſe:

Every Round Buſhel *with a plain and even Bottom, being made Eighteen* Inches *and a half wide throughout, and Eight* Inches *Deep, ſhould be eſteem'd a Legal* Wincheſter Buſhel, *according to the Standard in his* Majeſty's Exchequer.

Now a Veſſel being thus made will contain 2150,42 *Cubick Inches*, conſequently the *Corn Gallon* doth contain but $268\frac{4}{5}$ *Cubical Inches.*

Cub Inches		
268,8	= 1	*Gallon*
537,6	= 2 = 1	*Peck*
2150,4	= 8 = 4 = 1	*Buſhel*
17203,2	= 64 = 32 = 8 = 1	*Quarter.*

Note, { 4 *Buſhels* = a *Comb.* / 10 *Quarters* = a *Wey*, and / 12 *Weys* = a *Laſt of Corn.* }

I obſerv'd amongſt the Lead-Mines in *Derbyſhire*, (*Anno* 1692) that the *Miners* bought and ſold their Lead Ore, by a *Meaſure* which they call'd an Ore Diſh; whoſe Dimenſions I carefully took and found them

Thus { Length 21.3 / Breadth 6. / Depth 8.4 } *Inches.*

Conſe-

Consequently its Content is 1073,52 *Cubick Inches*, which is very near Equal to 4 *Corn Gallons*, according to the above-mentioned Settlement.

Nine of those Dishes they call a Load of *Ore*, which if it be pretty good, will produce about 3 hundred Weight of Lead.

8. *Of* Time.

It is not an easy Thing to give a true *Definition* of *Time*; for (according to the *Philosophick Poet*)

'Time *of it self is nothing, but from Thought*
'*Receives its Rise, by labouring Fancy wrought*
'*From Things consider'd, whilst we think on some*
'*As present, some as past, or yet to come.*
'*No Thought can think on* Time, *that's still confest,*
'*But thinks on Things in Motion or at Rest.*

And so on, Vide *Lucretius*, Book I.

That is, *Time* only shews the *Duration* or *Mutation* of Things, a Year being the *Standard* or *Integer*, by which such Continuance or Change is computed. And a *Year* is that *Space* of *Time* in which the *Sun* (apparently) compleats its *Revolution* from any one *Point* in the *Ecliptick* (an imaginary *Circle* in the *Heavens*) to the same *Point* again, which according to Modern *Observations* is perform'd in 365 *Days*, 5 *Hours*, 48 *Minutes*, 57 *Seconds*, 21 *Thirds*, &c. But a *Second* being the least part of *Time* that can be truly *Measured* by the *Motion* of any *Mechanical Engine*, as a *Clock*, &c. (a *Third* being less than the Twinkling of an Eye) I begin the following Table with *Seconds*.

Seconds "							
60 = 1	′ *Minute*						
3600 =	60 = 1	° *Hour*					
86400 =	1440 =	24 = 1	*Day*	°	′	"	
31556937 =	525949 =	8765 =	365 +	5 +	48 +	57 = 1	*Year, called a (Solar Year.*

But the common *Year*, usually call'd the *Julian Year*, doth consist of 365 *Days* and 6 *Hours*, and is divided into twelve unequal *Months*, called *Calendar Months*, whose Names and *Number* of *Days* are the Subject of every *Almanack*.

To

To these *Tables* it may not be amiss to give a brief Accou of such *Coins*, *Weights*, and *Measures* as are frequently mention in the Scriptures. As I have deduc'd them from those whi seem to be the most Correct, inserted in the *Index* to the lar *Bible*, Printed *Anno* 1702, and compared with those used *England*, by the Lord Bishop of *Peterborough*.

The *Hebrew Weights*, compared with	*Troy Weight.* Oz.	Pw.	Gra
A Gerah =	0 .	0 .	10 19/20
10 *Gerahs* = *a Bekah* =	0 .	4 .	13 1/2
2 *Bekahs* = *a Shekel* =	0 .	9 .	3
100 *Shekels* = *a Menah* =	45 .	12 .	12

Note, A *Shekel* is said to be their Original *Weight*.

Their *Coin*	*English Coin.* l.	s.	d.	
A *Silver Menah* =	7 .	1 .	5 1/4	Weight 60 *Shekels*.
Talent of *Silver* =	357 .	11 .	10 1/2	Weight is 300 *Shekels*.
Talent of *Gold* =	5075 .	15 .	7 1/2	The same Weight men
The *Gold Dram* =	1 .	0 .	4	tioned in *Ez.* ii. 19.

The *Roman Money* mentioned in the *New Testament*.

A *Denarius*, or *Silver Penny* = 7 *d.* 3 *Farthings*.
Asses of Copper = 0 . 3 *Farthings*.
Assarium = 0 . 1 1/2 *Farthing*.
Quadrans = 0 . 3/4 of a *Farthing*.
A Mite = 0 . 1/3 of a *Farthing*.

Their *Long Measures*, compared with	*English Measure.* Yar.	Feet.	Inch. Pa
A Finger's Breadth =	0.	0 .	0,9
4 *Fingers* = *a Hand's Breadth* =	0.	0 .	3,6
2 *Hands* = *the least Span* =	0.	0 .	7,2
3 *Hands Breadth* = *the longest Span* =	0.	0 .	10,9
2 *Spans* = *the longest Cubit* =	0.	1 .	9,8
4 *Cubits* = *a Fathom* =	2.	1 .	3,5
6 *Cubits* = *Ezekiel's Reed* =	3.	1 .	11,3
400 *Cubits* = *a Stadium* =	243.	0 .	7,2
10 *Stadiums* = *a Mile* =	2432.	0 .	0
3 *Miles* = *a Parasang* =	7296.	0 .	0
Which is 4 *English Miles* and	256.		

The

their *Meaſures* of *Capacity*, compared with { *Engliſh Wine.* / *Gal. Pints Inch.*

A Cotyla=	0 . 0½	3,037
A Log=	0 . 0½	9,83
4 *Logs*=*a Cab*=	0 . 3 .	10,458
10 *Cotyla's*=*an Omer*=	0 . 6 .	1,5
3 *Cabs*=*a Hin*=	1 . 2 .	2,5
2 *Hins*=*a Seah*=	2 . 4 .	5,
3 *Seahs*=*an Epha*=	7 . 4 .	15,
10 *Epha's*=*a Chomer*=	75 . 5 .	5,625

Sect. 2. Addition of *Weights*, &c.

The foregoing *Tables* being ſo well underſtood, as that you can readily tell (without pauſing) how many *Units* of any one *Denomination*, do make one of the next *Superior Denomination* (*eſpecially in thoſe* Tables *as are moſt uſeful for your Buſineſs*) it will then be as eaſy to *Add*, or *Subſtract* them, as to *Add*, or *Subſtract whole Numbers*, due care being taken in placing all *Numbers* that are of one *Denomination* exactly underneath each other. That is to ſay, in *Money* place *Pounds* under *Pounds*, *Shillings* under *Shillings*, *Pence* under *Pence*, &c. Underſtand the like in *Weights* and *Meaſures*, &c. according to their ſeveral *Denominations*: Then in *Addition* obſerve this *Rule*.

Rule.

Always begin with thoſe Figures *of the Loweſt or Leaſt* Denomination, *and* Add *them all together into one* Sum, *then conſider how many of the next* Superior Denomination *are contained in that* Sum, *ſo many Units you muſt carry to the ſaid next* Superior Denomination *to be* Added *together with thoſe* Figures *that ſtand there*; *and if any thing* Remain *over or above thoſe* Units *ſo carried*, *that* Overplus *muſt be ſet down underneath its own* Denomination: *And ſo proceed on from one* Denomination *to another until all be finiſhed.*

Example in Coin.

Let it be required to *Add* 35 *l.* 14 *s.* 06 *d.* and 27 *l.* 02 *s.* 10 *d.* and 54 *l.* 13 *s.* 04 *d.* and 10 *l.* 17 *s.* 09 *d.* into one *Sum*.

The particular *Sums* being placed, as before directed, will ſtand as in the Margin following.

Then according to the *Rule*, I begin with the *Pence* (being here the loweſt or leaſt *Denomination*) and *Adding* them all together, I find their *Sum* to be 29 *d.* that is 2 *s.* and 5 *d.* (for

24 *d.*

24 *d.*=2 *s*, and 29=24—5) the 5*d.* I set down underneath its own *Denomination*, and carry the 2 *s.* to the Place of *Shillings*, *Adding* them and all the *Shillings* together, I find the *Sum* to be 48 *s. viz.* 2 *l.* 8 *s.* I set down the 8 *s.* underneath its own place of *Shillings*, and carry the 2 *l.* to the Place of *Pounds*, *Adding* them and all the *Pounds* together, I find their *Sum* is 128 *l.* consequently the *Total Sum* required is 128 *l.* 08 *s.* 05 *d.*

l.	*s.*	*d.*
35	14	06
27	02	10
54	13	04
10	17	05
128	08	05

Now, for as much as it often happens in keeping Books of *Accompts*, (and in other Buſineſs) that it is required to *Add* up large *Sums* of Money, conſiſting of 30, 40, or more ſeveral particular *Sums*, nay, perhaps filling up the whole Length of a Sheet of Paper, I humbly conceive in thoſe Caſes the beſt and eaſieſt way will be to part them into Parcels, not exceeding above 10 or 12 particular *Sums* in each Parcel; that done, Add together all the *Sums* of thoſe Parcels into one *Sum*, and that will be the *Total Sum* required.

Alſo to avoid the making of *Points*, or other *Marks* amongſt your *Figures*, it will be convenient to get the following *Tables* by heart.

The Pence Table.

d.	*s.*	*d.*	*s.*
12	1	72	6
24	2	84	7
36	3	96	8
48	4	108	9
60	5	120	10

The Shillings Table.

s.	*l.*	*s.*	*l.*
20	1	120	6
40	2	140	7
60	3	160	8
80	4	180	9
100	5	200	10

The Uſe of theſe *Tables* is ſo obvious, that I preſume 'tis needleſs to explain them.

Example in Addition *of* Weights.

Troy Weight.

lb	*Oz.*	*Pw.*	*Gr.*
3	09	00	10
5	08	15	21
10	10	12	22
0	11	19	23
Sum 21	04	09	04

Averdupois Weight.

Tun.	*C.*	*Q.*	lb	*Oz.*
12	15	2	24	12
7	10	3	21	15
0	18	1	14	11
1	19	3	27	15
Sum 23	05	0	05	05

Example

Examples in *Addition* of *Long-Meaſure.*

rds	*Qrs.*	*Nails*		*Miles*	*Fur.*	*Poles*	*Yards*	*Feet*	*Inch.*
5	2	3		2	6	32	4	2	9
7	3	1		0	7	27	3	1	10
9	1	2		1	3	39	1	2	11
2	3	2	Sum	5	2	19	5	0	0

I think it needleſs to ſet down more *Examples* of this kind; if theſe 5 (eſpecially the laſt) be well underſtood, they will be ſufficient to ſhew how any other may be performed.

Sect. 3. Subſtraction *of Weights*, &c.

Subſtraction is but the Converſe of the precedent Work, and may be performed by obſerving this *Rule.*

Rule.

Begin with the Loweſt or Leaſt Denomination (*as before in* Addition) *and Take or* Subſtract *the* Figure (*or* Figures) *in that place of the* Subtrahend, *from the* Figure (*or* Figures) *that ſtand over them of the ſame* Denomination; *ſetting down the* Remainder (*as in* Page 12.) *But if that cannot be done, then you muſt increaſe the upper* Figure (*or* Figures) *with one of the next Superior* Denomination, *and from that* Sum *make* Subſtraction; *and ſo proceed to the next Superior* Denomination, *where you muſt pay the one borrowed, by adding* Unity *to the* Subtrahend *in that place*, &c. *as in whole* Numbers.

Examples in Coin.

	l.	*s.*	*d.*
From	386	09	08
Take	173	04	06
Remains	213	05	02

	l.	*s.*	*d.*
From	569	10	06
Subſt.	389	15	08
	179	14	10

The Firſt of theſe *Examples* is ſelf-evident. In the Second *Example*, beginning at the place of *Pence* (being here the Leaſt *Denomination*) I am to take 8 *d.* from 6 *d.* but becauſe that cannot be done, I muſt (according to the *Rule*) borrow one of the next *Denomination*, viz. 1 *s.* and *Add* it to the 6 *d.* which makes it 18 *d.* (for 1 *s.* = 12 *d.* and 12 *d.* + 6 *d.* = 18 *d.* then I take 8 *d.* from that 18 *d.* and there *Remains* 10 *d.* to be ſet down underneath the place of *Pence*; that done, I proceed to the place of *Shillings*, where I muſt now pay the 1 *s.* ſaying I borrowed, and 15 makes 16 from 10 cannot be, but

16 from 30 and there *Remains* 14. That is, I borrow one of the next *Denomination*, viz. 1 *l.* and Add to it the 10 *s.* which makes it 30 *s.* (for 1 *l.*=20 *s.* and 20+10=30) having ſet down the *Remaining* 14 *s.* underneath its own place of *Shillings*, proceed to the place of *Pounds*, where paying the 1 *l.* borrowed it will be 1 borrowed and 9 is 10 from 9 cannot be, but 10 from 19 and there *Remains* 9, and ſo on as in whole *Numbers* until all be finiſhed; And the *Remainder* will be 179 *l.* 14 *s.* 10 *d.*

This *Example* being a little conſidered will render all others in this *Rule* eaſy.

Examples in Weights.

	Troy Weight				*Averdupois Weight*			
	℔	*oz.*	*pwt.*	*gr.*	*c.*	*qr.*	℔	*oz.*
From	9	10	16	18	17	2	15	10
Take	5	09	18	22	14	3	18	12
	4	00	17	20	*Reſts* 2	2	24	14

Examples in Long Meaſure.

	yds	*qrs.*	*nails*		*miles*	*fur.*	*pol.*	*yds.*	*fect*	*inch*
From	78	3	2		22	3	26	3	1	11
Take	29	3	3		18	6	29	4	2	7
Reſts	48	3	3		3	4	36	4	0	10

Example in Time.

	days	°	′	″
From	27	18	35	21
Subſtract	16	21	46	36
Remains	10	20	48	45

The Proof of *Addition* and *Subſtraction* in theſe *Numbers* of different *Denominations*, is the very ſame with that of whole *Numbers* in *Page* 13. I ſhall therefore refer you to that place and omit repeating it here.

Sect. 4. *Of* Reduction.

By *Reduction*, *Numbers* of different *Denominations* are brought into one *Denomination*.

That is, it alters or changes any Superior *Denomination* propoſed, into any Inferior or Leſſer *Denomination* Required; ſuch

l keeping them Equivalent in value. And by that means they come fitly prepared for *Multiplication* and *Diviſion*; which herwiſe could not ſo conveniently be performed. Therefore the ſineſs of *Reduction* is very uſeful in the *Rule of Proportion*, ommonly called the *Golden Rule*, or *Rule of Three*) eſpecially thoſe who do not underſtand either *Vulgar* or *Decimal Fractions*. And it's thus performed:

Rule.

Conſider how many Units *of the* Denomination *Required, make one of that* Denomination *propoſed to be Reduced (which is eaſily known by its reſpective Table) and with that* Number *of* Units, Multiply *the* Denomination *propoſed, and their* Product *will be the* Number *Required.*

Example in Coin.

Let it be Required to Reduce or Change 357 *l.* into *Shillings*, and thoſe *Shillings* into *Pence*, which ſhall ſtill be Equal in value with the 357 *l.*

357
Multiply with 20 the *Shillings* in one *Pound*
7140=the *Shillings* in 357 *l.*
Multiply with 12 the *Pence* in one *Shilling*
1428
714
85680=the *Pence* in 357 *l.* as was Required.

Or 357 *l.* may be reduced into *Pence*, at one Operation: Thus,

357 *l.*
Multiply with 240 the *Pence* contained in one *Pound*
1428
714
85680=the *Pence* in 357 *l.* as before.

But when the *Numbers* propoſed to be Reduc'd are of ſeveral *Denominations*, and it is required to bring them all to the Loweſt; you muſt Reduce the higheſt or greateſt *Denomination* to the next leſs, Adding the *Numbers* that are of that leſs *Denomination* together, then *Reduce* their *Sum* to the next lower *Denomination*, Adding together all the *Numbers* that are of that *Denomination*, and ſo proceed gradually on until all is done.

Example.

Let it be required to Reduce 375 *l.* 17 *s.* 10 *d.* 3 *q.* into one *Denomination*, viz. into *Farthings*.

375 *l.* 17 *s.* 10 *d.* 3 *d.*

20

7500=the *Shillings* in 375 *l.*
+ 17 *s.*

7517=the *Shillings* in 375 *l.* 17 *s.*

12

15034
7517

90204=the *Pence* in 375 *l.* 17 *s.*
+ 10 *d.*

90214=the *Pence* in 375 *l.* 17 *s.* 10 *d.*

4

360856=the *Farthings* in 375 *l.* 17 *s.* 10 *d.*
+ 3 *q.*

360859 *Farthings*=375 *l.* 17 *s.* 10 *d.* 3 *q.* as was required.

The Work of this *Example*, and all other Operations of this kind, may be somewhat shortned by observing the following Method.

375 *l.* 17 *s.* 10 *d.* 3 *q.*

20 *Multiply* and *Add* in the 17 *s.*

7517

12 *Multiply* and *Add* in the 10 *d.*

15034
7517

90214

4 *Multiply* and *Add* in the 3 *qrs.*

360859 the *Farthings* as before.

Example in Troy Weight.

Suppose it be Required to *Reduce* 29 ℔ 8 *oz.* 18 *pwt.* 21 *gr.* into the Least *Denomination*, viz. into *Grains*.

Thus,

Thus, 29 ℔ 8 *oz.* 18 *pwt.* 21 *gr.*
ltiply with 12 the *oz.* in 1 ℔ and *Add* in the 8 *oz.*

66
29

356=the *Ounces* in 29 ℔ 8 *oz.*
ultiply with 20 the *pwts.* in 1 *oz.* and *Add* in the 18 *pwt.*

7138=the *pwts.* in 29 ℔ 8 *oz.* 18 *pwt.*
ultiply with 24 the *Gra.* in 1 *pwt.* and *Add* in the 21 *gr.*

28553
14278

171333 the *Grains*=29 ℔ 8 *oz.* 18 *pwt.* 21 *gr.*

Theſe Two *Examples* at Large being well underſtood, may ſuffice to ſhew how all Operations of this kind are performed; either in *Weights*, *Meaſures*, or *Time*. I ſhall only inſert a few *Examples* of each ſort for the Learner's practice.

1. In 23 *C.* 3 *qrs.* 21 ℔ 9 *oz.* *Averdupois Weight*; How many *Ounces*. *Anſwer* 42905 *Ounces*.

2. In 252 *Engliſh Miles*, How many *Yards*, *Feet*, and *Inches*. *Anſwer* 443520 *Yards*=1330560 *Feet*=15966720 *Inches*.

3. In 1692 common *Years*; How many *Days*, *Hours*, and *Minutes*. *Anſwer* 618003 *Days*, 14832072 *Hours*, 889924320 *Minutes*.

Note, a common *Year*=365 *Days*, 6 *Hours*, ſee *Page* 37.

4. In 5786 *Pounds*, 17 *Shillings*, 9 *Pence Sterling*: How many *Shillings*, *Pence*, and *Farthings*.

Anſwer, 115737 *s.* 1388853 *d.* or 5555412 *Farthings*. That is 5786 *l.* 17 *s.* 9 *d.*=115737 *s.* 9 *d.*=1388853 *d.* &c.

The next Thing will be to ſhew how to bring *Numbers* from a leſſer to a greater *Denomination*, which by moſt *Authors* is called (tho' very improperly)

Reduction *Aſcending.*

This Work is the Converſe of the laſt, and is performed by *Diviſion*. Thus,

Rule.

Conſider how many of the Denomination *propoſed make one of the* Denomination *required, and make that* Number *your* Diviſor, *by which* Divide *the* Denomination *propoſed; and the* Quotient *will be the* Number *required.*

Example

Example.

Let it be required to find how many *Shillings* and *Pounds* a contained in 85680 *Pence.*

The *Pence* in 1 *s.* are 12) 85680 (7140 *s.*=85680 *d.*

Again the *Shillings* in 1 *l.* are 20) 7140 (357 *l.* the *Answ* required.

Another Example in Coin.

How many *Pence*, *Shillings*, and *Pounds*, are contained 264859 *Farthings.*

12) 20)
4) 264859 (66214 *d.* (5517 *s.* (275 *l.*
24 62 151
08 21 117
05 94 (17) *s.*
19 (10) *d.*

Remains (3) *q.* { *Note*, the *Remainder* is always of th same *Denomination* with the *Dividen*

The laſt *Quotient* 275 *l.* together with the ſeveral *Remainder* gives the *Anſwer* required.

Viz. 275 *l.* 17 *s.* 10 *d.* 3 *q.*=264859 *Farthings*

Example in Troy Weight.

Suppoſe it were required to find how many *pwts. ozs.* and lb are contained in 171333 *Grains.*

20) 12)
24) 171333 *gr.* (7138 *pw.* (356 (29 lb
168··· 113 24
33 138 116
24 (18) *pwt.* 108
93 (8) *oz.*
72
213
192

Remains (21) *gr.*

Anſwer 29 lb 8 *oz.* 18 *pwt.* 21 *gr.* This and the laſt *Exampl* are the Reverſe or Proof of thoſe in *Page* 43, 45.

1. In 42905 *Ounces Averdupois Weight*; How many *Pounds* &c.

Thu

28) 4)
us 16) 42905 (2681 lb (95 *qrs.* (23 *C.*
109 252 15
130 161 (3)
25 140
(9) (21) *Answer* 23 *C.* 3 *qrs.* 21 lb 9 *oz.*

2. *In* 15966720 *Inches*; How many *English Miles*, &c.
Answer 252 *Miles*, &c. as Occasion requires.

There are many useful *Questions* may be answered by help of *Reduction* only: As the Changing of one sort of *Coin* for another; d comparing one sort of *Measure* with another, &c.

For Instance; Suppose one had 347 *Rix-dollars*, at 4*s.* 6*d.* *per ollar*; and desired to know how many *Pounds Sterling* they ke.

347
54=the *Pence* in one *Dollar*, viz. 4 *s.* 6 *d.*=54 *d.*
1388
1735 20)
12) 18738 *d.* (1561 *s.* (78 *l.*
67 161
73 (1)*s.*
18
(6)*d.*

Answer 78 *l.* 1 *s.* 6 *d. Sterling* are=347 *Rix-dollars.*

Quest. 2. *In* 645 *Flemish Ells*; How many *Ells English.*

Note, 3 *Quarters of a Yard English make one Ell Flemish,* d $1\frac{1}{4}$ *or* 5 *Quarters of a Yard is an English Ell.*

Therefore, 645
3=the *qrs* of a *Yard* in 1 *Ell Flemish.*
s. in 1 *Ell*=5) 1935 (387 *English Ells* for the *Answer.*

Quest. 3. Suppose a Bill of Exchange were accepted at *London*, r the Payment of 400 *l. Sterling*, for the Value deliver'd at *msterdam* in *Flemish* Money at 1 *l.* 13 *s.* 6 *d.* for one *Pound terling.* How much *Flemish* Money was delivered at *Amsterdam.*

First, 1 *l.* 13 *s.* 6 *d.*=402 *d.* the Value of one *Pound Sterling* t *Amsterdam.*

Then, 402 *d.*×400=160800 *d.*=670 *l. Flemish*, and so much as deliver'd at *Amsterdam.*

CHAP.

CHAP. IV.

Of Vulgar Fractions.

Sect. 1. *Of* Notation.

A *Fraction*, or *Broken Number*, is that which represents a *Part* or *Parts* of any thing proposed *(vide Page 3.)* and is expressed by Two *Numbers* placed one above the other with a Line drawn betwixt them:

Thus, $\frac{3}{4}$ *Numerator.* / *Denominator.*

The *Denominator* or *Number* placed underneath the *Line* denotes how many *Equal Parts* the thing is supposed to be *Divided* into (being only the *Divisor* in *Division.*) And the *Numerator* or *Number* placed above the *Line*, shews how many of those *Parts* are contained in the *Fraction* (it being the *Remainder* after *Division.) (See Page 29.)* And these admit of three Distinctions:

Viz. *Proper* or *Simple*, *Improper*, *Compound* } *Fractions.*

A proper, pure, or *Simple Fraction*, is that which is Less than an *Unit*. That is, it represents the immediate *Part* or *Parts* of any thing less than the whole, and therefore its *Numerator* is always Less than the *Denominator*.

As $\frac{1}{4}$ is one *Fourth Part*. $\frac{1}{3}$ is one *Third Part*, And $\frac{1}{2}$ is one *Half*. $\frac{2}{3}$ is two *Thirds*, &c.

An Improper *Fraction* is that which is greater than an *Unit*. That is, it represents some *Number* of *Parts* greater than the whole thing; And its *Numerator* is always greater than the *Denominator*.

As $\frac{5}{3}$ Or $\frac{9}{7}$ Or $\frac{41}{15}$ &c.

A Compound *Fraction* is a *Part* of a *Part*, consisting of several *Numerators* and *Denominators* connected together with the Word [of].

As $\frac{1}{3}$ of $\frac{3}{4}$ of $\frac{2}{5}$, &c. and are thus read, The one *Third* of the three *Fourths* of the two *Fifths* of an *Unit*.

That is, when a *Unit* (or whole thing) is first *Divided* into any *Number* of *Equal Parts*, and each of those *Parts* are

Subdivided

divi[illegible] ther *Parts*, and so on: Then those last *Parts* calle[illegible] *und Fractions*, or *Fractions of Fractions*.

As fo[illegible], suppose a *Pound Sterling* (or 20*s.*) be the *it* or [illegible] then is 8*s.* the $\frac{2}{5}$ of it, and 6*s.* the $\frac{3}{4}$ of those *Fi*[illegible] 2*s.* is the $\frac{1}{3}$ of those three *Fourths*. *Viz.* 2 *s.* of $\frac{1}{4}$ [illegible] ne *Pound Sterling*.

All *Co*[illegible] *Fractions* are reduced to single ones, Thus,

Rule.

Multi[illegible] *he* Numerators *into one another for a* Numerator, *all* [illegible] ominators *into one another for the* Denominator.

T[illegible] $\frac{1}{3}$ of $\frac{3}{4}$ of $\frac{2}{5}$ will become $\frac{6}{60}$. Or $\frac{1}{10}$.

1×[illegible] he *Numerator*, and 3×4×5=60 the *Denomina-* but [illegible] of a *Pound Sterling* is 2 *s.* As above.

---

ect. [illegible] **lter** or **Change** different **Fractions** into one [illegible] *ination* relating to the same Value.

n or[illegible] a clear Understanding of this *Section*, it will conv[illegible] premise this *Proposition*, *viz.* If a *Number* *ltip*[illegible] *Numbers* produce other *Numbers*, the *Numbers* duced [illegible] shall be in the same Proportion that the *mbers Mul*[illegible] are, 17 *Euclid* 7.

That is to say, if both the *Numerator* and *Denominator* of *Fraction* be Equally *Multiplied* into any *Number*, their *ducts* will retain the same Value with that *Fraction*.

As in these, $\frac{2\times 3}{3\times 3}=\frac{4}{6}$. Or $\frac{2\times 3}{3\times 3}=\frac{6}{9}$. Or $\frac{2\times 5}{3\times 5}=\frac{10}{15}$, &c.

That is, $\frac{2}{3}$ and $\frac{4}{6}$. Or $\frac{2}{3}$ and $\frac{6}{9}$. Or $\frac{2}{3}$ and $\frac{10}{15}$ are of the same e in respect to the whole or *Unit*.

rom hence it will be easy to conceive how Two, or more *ctions* that are of Different *Denominations*, may be alter'd hang'd into others that shall have one common *Denominator*, still retain the same value.

xample. Let it be required to change $\frac{2}{3}$ and $\frac{3}{7}$ into Two *Fractions* that shall have one common *Denominator*, and Retain the same value.

ccording to the foregoing Proposition, if $\frac{2}{3}$ be Equal- *Multiplied* with 7, it will become $\frac{14}{21}$ *viz.* $\frac{2\times 7}{3\times 7}=\frac{14}{21}$. n, if $\frac{3}{7}$ be Equally *Multiplied* with 3, it will become [illegible]. $\frac{3\times 3}{7\times 3}=\frac{9}{12}$. And by this means I have obtained Two *Fractions* $\frac{14}{21}$ and $\frac{9}{21}$ that are of one *Denomination*, and same value with the Two first proposed, *viz.* $\frac{14}{21}=\frac{2}{3}$ d $\frac{9}{21}=\frac{3}{7}$.

And from hence doth arise the General Rule for bringing *Fractions* into one *Denomination*.

Rule.

Multiply all the Denominators *into each other for a* N *(and Common)* Denominator. *And each* Numerator *into all* Denominators *but its own, for New* Numerators.

Example. Let the propoſed *Fractions* be, $\frac{1}{3}$. $\frac{2}{5}$. $\frac{3}{4}$. and $\frac{6}{7}$.

Then by the *Rule*.

A new *Denominator* will be thus found.

3
5
15
4
60
7
420

And the new *Numerators* will be thus found.

1.	2.	3.	6
5	3	3	3
5	6	9	18
4	4	5	5
20	24	45	90
7	7	7	4
140.	168.	315.	360

Hence 420 is the common *Denominator*; And 140. 16 315. 360 are the new *Numerators*, which being placed Fracti wiſe are $\frac{140}{420}$. $\frac{168}{420}$. $\frac{315}{420}$. $\frac{360}{420}$ the New *Fractions* required.

That is, $\frac{140}{420}=\frac{1}{3}$. $\frac{168}{420}=\frac{2}{5}$. $\frac{315}{420}=\frac{3}{4}$. and $\frac{360}{420}=\frac{6}{7}$.

Sect. 3. *To bring mix'd* Numbers *into* Fractions, *and the contrary.*

Mix'd *Numbers* are brought into improper *Fractions* by following *Rule.*

Rule.

Multiply *the* Integers *or whole* Numbers *with the* Denominator *of the given* Fraction, *and to their* Product Add Numerator, *the Sum will be the* Numerator *of the* Fracti *required.*

Example. $9\frac{4}{5}$ by the *Rule* will become $\frac{49}{5}$. For $9\times5=\frac{49}{5}$.

And, $\frac{45}{5}+\frac{4}{5}=\frac{49}{5}$ the improper *Fraction* required.

Again, $13\frac{11}{15}$ will become $\frac{206}{15}$. For $13\times15=\frac{195}{15}$.

And $\frac{195}{15}+\frac{11}{15}=\frac{206}{15}$. And ſo for any other as occaſion requires.

To find the true Value of any improper *Fraction* given only the Converſe of this *Rule.* For if $\frac{49}{5}=9\frac{4}{5}$ as before evident

lent: Then it follows that if 49 be *Divided* by 5, the *otient* will give $9\frac{4}{5}$. And if 206 be *Divided* by 15 it will $13\frac{11}{15}$ *&c.* consequently it follows, That

f the Numerator *of any improper* Fraction *be* Divided *by its* nominator, *the* Quotient *will discover the true Value of that* ction.

Examples.

$\frac{35}{7}=5$. And $\frac{41}{9}=4\frac{5}{9}$. And $\frac{122}{20}=6\frac{1}{20}$. Or $\frac{25}{4}=3\frac{5}{4}$ *&c.*

When whole *Numbers* are to be expres'd Fraction-wise, it is giving them an *Unit* for a *Denominator*. Thus 45 is $\frac{45}{1}$. $\frac{9}{1}$. and 25 is $\frac{25}{1}$, *&c.*

Sect 4. *To* Abbreviate *or* Reduce Fractions *into their Lowest or Least* Denomination.

This is done, not out of any Necessity, but for the more convenient managing of such *Fractions* as are either proposed in large Terms, or swell into such, either by *Addition* or otherwise: Besides it's most like an Artist to express or set down all *Fractions* in the lowest Terms possible; And to perform that, it will be necessary to consider of these following Propositions.

Numbers are either Prime or Composed.

1. A *Prime Number* is that which can only be *Measured* by an *Unit*. *Euclid* 7. *Defin.* 11.

That is, 3. 5. 7. 11. 13. 17. *&c.* are said to be Prime *Numbers*, because 'tis not possible to *Divide* them into Equal Parts by any other *Number* but *Unity* or 1.

2. *Numbers* Prime the one to the other, are such as only an *Unit* doth *Measure*, being their common *Measure*. *Euclid* 7. *Defin.* 12.

For Instance, 7 and 13 are Prime *Numbers* to each other, because they cannot be divided by any *Number* but an *Unit*. And 9 and 14 are also Prime *Numbers* to each other, for altho' 3 will *Measure* or *Divide* 9 without leaving a *Remainder*, yet 3 will not *Measure* 14 without leaving a *Remainder*; Again, altho' 2 will *Measure* 14 without any *Remainder*, yet 2 will not *Measure* 9 without leaving a *Remainder*, &c.

3. A *Composed Number* is that which some certain *Number* *measureth*. *Euclid* 7. *Defin.* 13.

For Instance, 15 is a composed *Number* of 3 and 5, for $3\times5=15$, consequently 3 or 5 will justly *Measure* 15. Also 20

is composed of 5 and 4, *viz.* 5×4=20, therefore 5 and 4 will each justly *Measure* 20.

4. *Numbers* composed the one to the other, are they which some *Number* being a common *Measure* to them both doth *Measure*. *Euclid* 7. *Defin.* 14.

That is, If Two or more *Numbers* can be *Divided* by one and the same *Divisor*; then are those *Numbers* said to be composed one to another.

For Instance, 14 and 21 are *Numbers* composed the one to the other, because they can both be *Measured* or *Divided* by 7. For 7×2=14, and 7×3=21, therefore 7 is a common *Measure* to 14 and 21. So that if $\frac{14}{21}$ were proposed to be abbreviated it will become $\frac{2}{3}$.

Thus $\begin{cases} 7)14=2 \\ 7)21=3 \end{cases}$

And how those greatest common *Measures* may be found comes from *Euclid* 7. *prob.* 1, 2, 3, and is thus:

Rule.

Divide *the greater* Number *by the lesser*, *and that* Divisor *by the* Remainder *(if there be any) and so on continually until there be no* Remainder *left: Then will that last* Divisor *be the greatest Common* Measure *(and if it happen to be* 1, *then are those* Numbers *Prime* Numbers, *and are already in their Lowest Terms, but if otherwise)* Divide *the* Numbers *by that last* Divisor, *and their* Quotient *will be their Least Terms required.*

Example.

Let it be required to find the greatest common *Measure* of 72 and 108, *viz.* Of $\frac{72}{108}$.

72) 108 (1
72

36) 72 (2
72

(0)

Here because there's no *Remainder*, 36 is the greatest common *Measure*.

Therefore, $\begin{cases} 36)72=2 \\ 36)108=3 \end{cases}$ Hence $\frac{72}{108}$ is Abbreviated to $\frac{2}{3}$ the lowest *Terms*.

Again, To find the greatest common *Measure* of 744 and 899

Thus

Thus, 744) 899 (1
744
155) 744 (4
620
124) 155 (1
124
31) 124 (4
124
(0)

Here 31 is found to be the greateſt common *Meaſure* by which 744 and 899 may be Abbreviated to 24 and 29 their Loweſt *Terms*.

Thus, $\frac{31}{31}) \frac{744}{899} (=\frac{24}{29}$. *&c*

Note, *If the propoſed* Numbers *be even, they may be brought lower by a continued Halving of them, ſo long as as they can be Halved*, viz. Divided *by* 2.

Example.

'Tis required to *Reduce* $\frac{56}{84}$ to its leaſt *Terms*.

Firſt, $\frac{2}{2}) \frac{56}{84} (=\frac{28}{42}$. Again, $\frac{2}{2}) \frac{28}{42} (=\frac{14}{21}$.

This done, you may eaſily perceive that 7 will be the common *Meaſure* to 14 and 21, *viz.* $\frac{7}{7}) \frac{14}{21} (=\frac{2}{3}$. *&c.*

If the *Numbers* propoſed to be *Reduced* have each a *Cypher*, or *Cyphers Annexed* to them, they will be Abbreviated by cutting off a like *Number* of *Cyphers* from both.

Thus, $\frac{150}{300}$ will be $\frac{15}{30}$. And $\frac{200}{300}$ will be $\frac{2}{3}$, *&c.*

That is, $\frac{150}{300}=\frac{15}{30}=\frac{1}{2}$. And $\frac{200}{300}=\frac{2}{3}$. And $\frac{360}{400}=\frac{36}{40}=\frac{18}{20}=\frac{9}{10}$.

Sect. 5. Addition *of* Fractions.

What hath been done by the *Rules* in this *Chapter*; is chiefly to prepare and fit *Fractions* of different *Denominations* for *Addition* or *Subſtraction*, as Occaſion requires, *viz.* If they are *Compound Fractions*, they muſt be *Reduced* to Simple or Pure *Fractions*, *per Rule*, *Sect.* 1.

If they are of different *Denominations*, they muſt be altered or chang'd, *per Rule*, *Sect.* 2.

That is, all *Fractions* muſt be brought into one *Denomination* before they can either be *Added* or *Subſtracted*, and that being done, *Addition* is thus performed.

Rule.

Add *together all the* Numerators, *and their* Sum *will be a* New Numerator, *under which* Subſcribe *the* Common Denominator.

Examples

Examples in Simple Fractions.

Let it be proposed to *Add* $\frac{1}{3}$, $\frac{2}{5}$ and $\frac{3}{4}$ together. First, $\frac{1}{3}=\frac{20}{60}$. $\frac{2}{5}=\frac{24}{60}$, and $\frac{3}{4}=\frac{45}{60}$. *per Sect.* 2.

Then $\frac{20}{60}+\frac{24}{60}+\frac{45}{60}=\frac{89}{60}$. the *Sum* required, which according to *Section* 3. is $1\frac{29}{60}$. *viz.* $\frac{89}{60}=1\frac{29}{60}$.

Examples in Compound Fractions.

Let it be required to *Add* $\frac{3}{7}$ and $\frac{2}{3}$ of $\frac{3}{4}$ into one *Sum*. First $\frac{2}{3}$ of $\frac{3}{4}$ becomes $\frac{6}{12}$ or $\frac{1}{2}$ *per Sect.* 1. And (*per Sect.* 2.) $\frac{3}{7}$ and $\frac{1}{2}$ is $\frac{6}{14}$ and $\frac{7}{14}$. *viz.* $\frac{3}{7}=\frac{6}{14}$ and $\frac{1}{2}=\frac{7}{14}$ but $\frac{7}{14}+\frac{6}{14}=\frac{13}{14}$ the *Sum Required*, viz. $\frac{3}{7}+\frac{2}{3}$ of $\frac{3}{4}=\frac{13}{14}$.

Examples in Mix'd Numbers.

'Tis required to *Add* $5\frac{2}{3}$ to $7\frac{3}{4}$. these *per Sect.* 3. will be $'\frac{7}{3}$ and $^3\frac{1}{4}$. But $'\frac{7}{3}$ and $^3\frac{1}{4}$ will become $\frac{68}{12}$ and $\frac{93}{12}$ *per Sect.* 2. Then $\frac{68}{12}+\frac{93}{12}=\frac{161}{12}$, and $\frac{161}{12}=13\frac{5}{12}$. the *Sum Required*.

Or you may bring only the *Fractions* to one *Denimination*, Thus, $5\frac{2}{3}$ and $7\frac{3}{4}$ will become $5\frac{8}{12}$ and $7\frac{9}{12}$.

Then $5\frac{8}{12}+7\frac{9}{12}=12\frac{17}{12}$. That is, $13\frac{5}{12}$. As before.

Sect. 6. Substraction of Fractions.

Rule { Substract *one* Numerator *from the other (according as the Question requires) and their Difference will be a new* Numerator, *under which Subscribe the Common* Denominator, *as in* Addition.

Example 1.

Let it be required to take $\frac{2}{9}$ out of $\frac{3}{7}$ First $\frac{2}{9}$ and $\frac{3}{7}$. *per Sect.* 2. will become $\frac{14}{63}$ and $\frac{27}{63}$. then $\frac{27}{63}-\frac{14}{63}=\frac{13}{63}$, that is, $\frac{3}{7}-\frac{2}{9}=\frac{13}{63}$ As was required.

Example 2.

'Tis required to *Substract* $\frac{2}{3}$ of $\frac{8}{9}$ from $\frac{13}{24}$. First, $\frac{2}{3}$ of $\frac{8}{9}=\frac{16}{27}$. *per Sect.* 1. Again $\frac{16}{27}$ and $\frac{13}{24}$ will become $\frac{224}{378}$ and $\frac{351}{378}$. *per Sect.* 2. Then $\frac{351}{378}-\frac{224}{378}=\frac{127}{378}$.

Example 3.

From $6\frac{1}{8}$ *Substract* $3\frac{19}{48}$. First, $6\frac{1}{8}=\frac{49}{8}$. and $3\frac{19}{48}=\frac{163}{48}$ *per Rule Sect.* 3. Again, $\frac{49}{8}=\frac{2352}{384}$, and $\frac{163}{48}=\frac{1304}{384}$. *per Rule Sect.* 2. Then, $\frac{2352}{384}-\frac{1304}{384}=\frac{1048}{384}=2\frac{280}{384}=2\frac{35}{48}$. Or otherwise thus,

thus: First, $6\frac{1}{8}=5\frac{9}{8}$, then bring $\frac{9}{8}$ and $\frac{19}{48}$ into one *Denomination*, viz. $5\frac{9}{8}=5\frac{432}{384}$ and $3\frac{19}{48}=2\frac{152}{384}$.
Then $5\frac{432}{384}-3\frac{152}{384}=2\frac{280}{384}=2\frac{35}{48}$. As before.

Example 4.

Let it be required to *Substract* $\frac{3}{7}$ of $\frac{5}{9}$ of $\frac{2}{3}$ from 7.
First, $\frac{3}{7}$ of $\frac{5}{9}$ of $\frac{2}{3}=\frac{30}{189}$. And $7=6\frac{189}{189}$.
Then $6\frac{189}{189}-\frac{30}{189}=6\frac{159}{189}=6\frac{53}{63}=7-\frac{3}{7}$ of $\frac{5}{9}$ of $\frac{2}{3}$. As was required.

If these few *Examples* be well understood, the whole Business of *Adding* and *Substracting Vulgar Fractions* will be easy; which is really much more difficult to perform than either *Multiplication* or *Division*, as will appear in the next *Section*.

Sect. 7. Multiplication of Fractions.

In order to perform either *Multiplication* or *Division*, you must prepare the *Terms* to be *Multiplied* (or *Divided*) thus; Reduce *Compound Fractions* to *Simples Ones*, *per Sect.* 1. Bring *mix'd Numbers* into improper *Fractions*, and express *Whole Numbers Fractionwise*, *per Sect.* 3. Also it will be convenient to Abbreviate them to their smallest *Terms* when it can be done. Then *Multiplication* may be thus performed.

Rule. Multiply *the* Numerators *one into another for a* New Numerator; *and the* Denominators *one into another for a* New Denominator. *As in these*

Examples.

1. The *Product* of $\frac{2}{5}$ into $\frac{3}{7}=\frac{6}{35}$. That is, $\frac{2\times3}{5\times7}=\frac{6}{35}$.
2. And the *Product* of $\frac{9}{18}$ into $\frac{20}{27}=\frac{180}{432}$. Or $\frac{5}{12}$.
3. Again, the *Product* of $\frac{7}{11}$ into $\frac{2}{5}$ of $\frac{5}{7}=\frac{70}{385}$. Or $\frac{2}{11}$.
For $\frac{2}{5}$ of $\frac{5}{7}=\frac{10}{35}$. Then $\frac{7}{11}\times\frac{10}{35}=\frac{70}{385}=\frac{2}{11}$.
4. Let it be Required to *Multiply* 6 with $3\frac{2}{5}$. These prepared for the Work will stand thus. $\frac{6}{1}\times\frac{17}{5}$.
viz. $6=\frac{6}{1}$ and $3\frac{2}{5}=\frac{17}{5}$. Then $\frac{6}{1}\times\frac{17}{5}=\frac{102}{5}$; or $20\frac{2}{5}$.
Or, otherwise thus, $6\times3=18$. And $\frac{2}{5}\times6=\frac{12}{5}=\frac{2}{5}$.
Then $18+2\frac{2}{5}=20\frac{2}{5}$. As before.
5. Let it be Required to *Multiply* $7\frac{4}{9}$ with $5\frac{3}{7}$.
First $7\frac{4}{9}=\frac{67}{9}$. and $5\frac{3}{7}=\frac{38}{7}$. Then $\frac{67}{9}\times\frac{38}{7}=\frac{2546}{63}=40\frac{26}{63}$.

Now the Reason of this *Rule* for *Multiplying* of *Fractions*, and consequently of these Operations, and all other performed by it; will be evident from this following.

Viz.

Viz. If $\frac{4}{2}$ be *Multiplied* with $1\frac{2}{3}$ according to the *Rule*, their *Product* will be $4\frac{8}{6}$. But $4\frac{8}{6}=8$.

Now $\frac{4}{2}=2$. and $1\frac{2}{3}=4$ *per Sect.* 3. But $4\times2=8$. *Ergo* &c.

Sect. 8. Division of Fractions.

The *Fractions* being first prepared as before directed, *Division* may be thus performed:

Rule { Multiply *the* Numerator *of the* Dividend *into the* Denominator *of the* Dividing Fraction *for a New* Numerator: *And* Multiply *the other* Numerator *and* Denominator *together for a New* Denominator.

Examples.

1. Let $\frac{6}{35}$ be *Divided* by $\frac{3}{7}$. *viz.* $\frac{3}{7})\frac{6}{35}(\frac{42}{105}=\frac{2}{5}$ the *Quotient*.
That is, according to the *Rule* $6\times7=42$ the new *Numerator*, and $35\times3=105$, the new *Denominator*, &c. as above.

2. Let it be requir'd to *Divide* $\frac{20}{27}$ by $\frac{5}{12}$. *viz.* $\frac{5}{12})\frac{20}{27}(\frac{240}{135}=1\frac{7}{9}$.
For $12\times20=240$ the new *Numerator*, and $27\times5=135$ the new *Denominator*, &c. as before.

3. Suppose it were required to *Divide* $\frac{2}{11}$ by $\frac{2}{5}$ of $\frac{5}{7}$.
First, $\frac{2}{5}$ of $\frac{5}{7}=\frac{10}{35}$ Then $\frac{10}{35})\frac{2}{11}(\frac{70}{110}=\frac{7}{11}$.

4. Let $20\frac{2}{5}$ be *Divided* by $3\frac{2}{5}$ *viz* $\frac{102}{5}$ by $\frac{17}{5}$.
For $20\frac{2}{5}=\frac{102}{5}$, and $3\frac{2}{5}=\frac{17}{5}$, Then $\frac{17}{5})\frac{102}{5}$ (=6 the *Quotient*.

5. Let it be Required to *Divide* $40\frac{26}{63}$ by $5\frac{3}{7}$.
First, $40\frac{26}{63}=2\,5\,46\frac{6}{63}$, and $5\frac{3}{7}=3\,8\frac{8}{7}$. Then $3\,8\frac{8}{7})2\,5\,46\frac{6}{63}(1\,7\,8\,2\,2\frac{4}{9}$.
But $17822\frac{4}{9}=7\frac{4}{9}$ the true *Quotient* Required.

6. Suppose it were Required to *Divide* 13 by $\frac{5}{7}$.
First, $13=\frac{91}{7}$. Then $\frac{5}{7})\frac{91}{7}(\frac{91}{35}=18\frac{1}{5}$, the *Quotient*.

7. Again, let it be Required to *Divide* $\frac{5}{7}$ by 6.
Viz. $\frac{6}{1})\frac{5}{7}(\frac{5}{42}$ for the *Quotient* Required.

N. B. *From hence you may observe, that when any* Whole Number *is* Divided *by a* Fraction *Less than* Unity *or* 1, *the* Quotient *will be Greater than the* Number *propos'd to be* Divided: *But if any* Fraction *be* Divided *by a* Whole Number, *greater than* 1, *Then the* Quotient *will be Less than the* Dividend: *As in the Two last* Examples.

As

As to the *Reaſon* (or *Proof*) of this *Rule* for *Dividing Fractions*: 'Tis only the *Converſe* to that of *Multiplication*, and will be very evident from this following.

Let $\frac{48}{6}$ be *Divided* by $\frac{4}{2}$. Which according to the *Rule* is thus, $\frac{4}{2}$) $\frac{48}{6}$ ($\frac{96}{24}$=4. The true *Quotient*. Now $\frac{48}{6}$=8. And $\frac{4}{2}$=2. *per Sect.* 3. Conſequently $\frac{48}{6}$ *Divided* by $\frac{4}{2}$ is but the ſame with 8 *Divided* by 2. *viz.* 2.) 8 (4. The *Quotient* as before.

I could have inſerted *Geometrical Demonſtrations*, for the *Rules* of *Multiplication* and *Diviſion* of *Fractions*; but ſuppoſing the Learner purely unacquainted with thoſe kind of Demonſtrations, I thought theſe might be more intelligible to him, eſpecially in this place.

CHAP. V.

Of Decimal Fractions.

WHEN, or by whom this Excellent Invention of Decimal Arithmetick, *was Firſt introduced is uncertain, but doubtleſs its Improvements, and the Perfection it's now in, is owing to latter Years.*

Sect. 1. *Of* Notation.

In *Decimal Fractions*, the *Integer* or *Whole Thing* (whether it be *Coin*, *Weight*, *Meaſure*, or *Time*, &c.) is ſuppos'd to be *Divided* into *Ten Equal Parts*; and every one of thoſe *Ten Parts* are ſuppoſed to be *Subdivided* into other *Ten Equal Parts*, &c. *ad infinitum*.

The *Integer* being thus *Divided (by Imagination)* into 10, 100, 1000, 10000, &c. *Equal Parts*, becomes the *Denominator* to the *Decimal Fraction*.

Thus, $\frac{2}{10}$. $\frac{3}{100}$. $\frac{7}{1000}$. $\frac{53}{10000}$. $\frac{754}{100000}$. &c.

Now theſe *Denominators* are ſeldom or never ſet down, but only the *Numerators*; and thoſe are either diſtinguiſhed, or ſeparated from *Whole Numbers* by a *Point* or a *Comma*.

Thus, 5,4 is $5\frac{4}{10}$. and 0,7 is $\frac{7}{10}$. 35,05 is $35\frac{5}{100}$, &c.

But before we proceed further in *Notation*, it will be convenient for the Learner to conſider of the following *Table*, (taken out of the Learned Mr. *Oughtred's Clavis Mathematica*) which ſhews the very Foundation of *Decimal Fractions*.

Whole Numbers							Decimal Parts						
5	4	3	2	1	0,	1	2	3	4	5	6	&c.
												Parts of a Million.
											Parts of 100 Thousands.	
										Parts of Ten Thousands.		
									Parts of a Thousand			
								Parts of a Hundred.				
							Parts of Ten, or $\frac{1}{10}$.					
						Units Place.						
					Tens.							
				Hundreds.								
			Thousands.									
		Tens of Thousands.										
&c.												

By this *Table* it is evident, that as in whole *Numbers* or *Integers*, every *Degree* from the *Units* Place increaſes towards the left-hand by a Ten-fold *Proportion*: So in *Decimal Parts* every *Degree* is decreaſed towards the right-hand by the ſame *Proportion*, viz. by *Tens*.

Therefore theſe *Decimal Parts* or *Fractions*, are really more *Homogeneal*, or agreeing with *Whole Numbers*, than *Vulgar Fractions*; for indeed all plain *Numbers* are in effect but *Decimal Parts* one to another.

That is, ſuppoſe any *Series* of *Equal Numbers*, as 444, &c. The firſt 4 towards the Left is *Ten* times the *Value* of the 4 in the middle, and that 4 in the middle is *Ten* times the *Value* of the laſt 4 to the Right of it, and but the *Tenth Part* of that 4 on the Left, &c.

Therefore all or any of them may be taken either as *Integers* or *Parts* of an *Integer*: If *Integers*, then they muſt be ſet down without any *Comma* or Separating *Point* betwixt them thus, 444. But if *Integers*, and one *Part* or *Fraction*, put a *Comma* betwixt them thus, 44,4 which ſignifies 44 *Whole Numbers*, and 4 *Tenths* of an *Unit*: Again, if two Places of *Parts* be required, ſeparate them with a *Comma* thus, 4,44 *viz.* 4 *Units*, and 44 *Hundredth Parts* of an *Unit*, &c.

From hence (duly compared with the *Table*) it will be eaſy to conceive that *Decimal Parts* take their *Denomination* from the Place of their laſt *Figure*.

That is, $\left\{\begin{array}{l} ,5=\frac{5}{10} \\ ,56=\frac{56}{100} \\ ,056=\frac{56}{1000} \end{array}\right\}$ *Parts* of an *Unit*, &c.

Cypher

Cyphers annex'd to *Decimal Parts*, alter not their *Value*. As, ,50, and ,500, or 5000, &c. are each but 5 *Tenths* of an *Unit*, For $\frac{50}{100}=\frac{5}{10}$. And $\frac{500}{1000}=\frac{5}{10}$. Or $\frac{5000}{10000}=\frac{5}{10}$. *Per Sect.* 4. of the laſt *Chapter*.

But *Cyphers* prefix'd to *Decimal Parts* decreaſe their *Value*, by removing them further from the *Comma*.

Thus,
,5= 5 *Tenth Parts*.
,05= 5 *Parts* of a *Hundred*.
,005= 5 *Parts* of a *Thouſand*.
,0005= 5 *Parts* of *Ten Thouſand*, &c.

Conſequently the true *Value* of all *Decimal Parts* are known by their Diſtance from the *Units* Place; the which being once rightly underſtood, the reſt will be eaſy.

Sect. 2. Addition, *and* Subſtraction *of* Decimals.

In ſetting down the propoſed *Numbers* to be *Added*, or *Subſtracted*, great care muſt be taken in placing every *Figure* directly underneath thoſe of the ſame *Value*, whether they be *Mix'd Numbers*, or Pure *Decimal Parts*, and to perform that you muſt have a due regard to the *Comma's*, or ſeparating *Points*, which ought always to ſtand in a direct Line one under another; and to the Right-hand of them carefully place the *Decimal Parts*, according to their reſpective *Values*, or *Diſtances* from *Unit*. Then

Rule. Add, *or* Suſtract *them as if they were all* Whole Numbers; *and from their* Sum *or* Difference, *cut off ſo many* Decimal Parts *as are the moſt in any of the given* Numbers.

Examples in Addition.

Let it be required to find the *Sum* of theſe following *Numbers*, viz. 34,5+65,3+128,7+95+87,8+7,9, which being truly placed, will ſtand

Thus,
34,5
65,3
128,7
95,0
87,8
7,9

Their *Sum* required, 419,2

Example 2.

Let it be required to find the *Sum* of 25,854+34,578+9,076+13,907.

25,854
34,578
9,076
13,907

83,415 The *Sum* required.

When the *Decimal Parts* propos'd to be *Added* (or *Substracted*) have not the same *Number* of Places, you may for convenience of Operation supply or fill up the void Places, by Annexing *Cyphers*. As in these *Examples*.

Example 3.	*Example* 4.	*Example* 5.
45,0700	574,678953	0,975642
50,7580	95,796430	,745257
123,0057	78,054600	,000598
74,7020	54,789000	,800700
24,8000	8,900000	,640530
318,3357	*Sum* 812,218983	3,162727

Examples in Substraction.

Let it be Required to find the Difference between 45,375 and 74,284.

	Example 1.	*Example* 2.	*Example* 3.
That is,	From 74,284	From 437,5	From 75,0034
	Take 45,375	Take 89,657	Take 57,875
	Remains 28,909	347,843	*Remains* 17,1284

Example 4.

Let it be Required to find the Excess between 562 and 93,5784.

	Example 4.	*Example* 5.
That is,	From 562,	From 345,7578
	Take 93,5784	Take 157,
	the *Excess* 468,4216	188,7578

Note, The two last *Examples* are supposed to be supply'd with *Cyphers*, which if actually done would stand thus,

562,0000		345,7578
93,5784		157,0000
Remains 468,4216	As before,	188,7578

Exampl

Example 6.		*Example* 7.	
From	0,547893	From	1,000000
Take	0,439758	Take	0,997543
	0,108135		0,002457

The *Proof* of *Addition*, and *Subſtraction* in *Decimals* is the ſame with that of *Whole Numbers*, Page 13, &c.

Sect. 3. Multiplication *of* Decimals.

Whether the *Factors* or *Numbers* to be *Multiplied* are pure *Decimals*, or *Mix'd*. *Multiply* them as if they were all *Whole Numbers*, and for the true *Value* of their *Product* obſerve this *Rule*.

Rule. *Cut off* (viz. *ſeparate with a* Comma) *ſo many Places of* Decimal Parts *in the* Product, *as there are in both the* Factors *accounted together*. *As in theſe*.

Example 1.	*Example* 2.
3,024	32,12
2,23	24,3
9072	9636
6048	12848
6048	6424
6,74352	780,516

The Reaſon why ſuch a *Number* of *Decimal Parts* muſt be cut off in the *Product*, may be eaſily Deduced from theſe *Examples*. Thus,

In *Example* 1. 'Tis evident, that 3 the whole *Number* in the *Multiplicand*, being *Multiplied* with 2, the whole *Number* in the *Multiplier*; can produce but 6 (*viz.* 3×2=6) So that of neceſſity all the other *Figures in the Product* muſt be *Decimal Parts*; according as the *Rule* directs.

Or, the *Rule* is evident from the *Multiplication* of *Whole Numbers* only: Thus, ſuppoſe 3000 were to be *Multiplied* with 200, their *Product* will be 600000; That is, there will be ſo many *Cyphers* in the *Product*, as are in both the *Factors*, (*Vide Page* 18.) Now if inſtead of thoſe *Cyphers* in the *Factors*, we ſuppoſe the like *Number* of *Decimal Parts*; then it follows, that there ought to be the ſame *Number* of *Decimal Parts* in the *Product*, as there were *Cyphers* in the *Factors*.

Again, the *Rule* may be otherwiſe made evident from *Vulgar Fractions*, thus: Let 32,12 be *Multiplied* with 24,3, and

and their *Product* will be 780,516 as in *Example* 2. above. Now 32,12=$32\frac{12}{100}$. and 24,3=$24\frac{3}{10}$. which being brought into Improper *Fractions* (*per Sect.* 3. *Page* 50.) will become $32\frac{12}{100}=\frac{3212}{100}$. and $24\frac{3}{10}=\frac{243}{10}$.

Then $\frac{3212}{100}\times\frac{243}{10}=\frac{780516}{1000}$. *per Sect.* 7. *Page* 55.
But $\frac{780516}{1000}=780\frac{516}{100}$. *viz.* 780,516 as before.

Any of these Three Ways do, I presume, sufficiently prove the Truth of the abovesaid *Rule*, &c.

Example 3.	*Example* 4.
78,546	5745
436	,0675
471276	28725
235638	40215
314184	34470
34246,056	387,7875

N. B. *It sometimes falls out in* Multiplying *Parts with Parts, that there will not be so many* Figures *in the* Product, *as there ought to be places of* Decimal Parts *by the* Rule: *In that Case you must supply their Defect by prefixing* Cyphers *to the* Product; *as in these* Examples.

Example 5.	*Example* 6.
,2365	,0347
,2435	,0236
11825	2082
7095	1041
9460	694
4730	,00081892
,05758775	

When any proposed *Number* of *Decimals* is to be *Multiplied* with 10 . 100 . 1000 . 10000, &c. 'Tis only Removing the separating Point in the *Multiplicand*, so many places towards the Right-hand, as there are *Cyphers* in the *Multiplier*.

Thus, ,578 × 10=5,78. And, ,578 × 100=57,8
Again, ,578 × 1000=578. Or, ,578 × 10000=5780.

These

Theſe things being conſider'd, it will be eaſy to *Multiply Decimals*, and determine their true *Products*. As in theſe following *Examples*.

57,056 *Multiplied* into 0,578 will *Produce* 32,978368
7,6543 into 5,4246 will *Produce* 41,52151578
0,56879 × 0,05674=0,0322731446
0,03246 × 0,02364=0,0007672544
87649 × 0,03687=3231,61863
94,35786 × 6,57869=620,75111000034
3,141592 × 52,7438=165,6995001296

Now it oftentimes happens, that it will be needleſs to expreſs all the *Figures* of the *Product* at large, (eſpecially, when the *Factors* have each of them many places of *Decimal Parts*, as in the Two laſt *Examples*) only ſo many of them as may ſuffice for the intended Deſign; and yet the *Product* may be as true to ſo many Figures as are retained, as if the *Factors* had been *Multiplied* at Large. And ſuch Compendious Contractions are not only of curioſity, but may alſo be found of great eaſe and uſe to the Ingenious Practitioner; eſpecially in *Reſolving Adfected Æquations*, or in calculating of *Trigonometrical Problems* by the *Natural Sines* and *Tangents*, &c. All which may be thus perform'd.

Viz. *Set the* Units *place of the* Multiplier *directly underneath that* Figure *of the* Multiplicand, *whoſe place you intend to keep in the* Product; *And place all the other* Figures *of the* Multiplier *in a quite contrary order to the uſual way. Then in* Multiplying *always begin at that* Figure *of the* Multiplicand *which ſtands over the* Figure *wherewith you are then a* Multiplying, *ſetting down the Firſt* Figure *of each particular* Product, *directly underneath one another; yet herein you muſt have a due regard to the Increaſe which would ariſe out of the Two next* Figures *to the Right-hand of that* Figure *in the* Multiplicand *which you then begin with.*

Example.

Let it be required to *Multiply* 3,141592 with 52,7438 and let there be only Four places of *Decimal Parts* retained in the *Product*.

If the propoſed *Numbers* were to be *Multiplied* at Large, they muſt ſtand in a direct order as uſual.

Thus

Thus { 3,141592 / 52,7438 } And would produce Ten places of parts, as in the laſt *Example.*

But being 'tis required to have only Four places of thoſe parts in the *Product*, ſet them down as above directed, and they will ſtand

Thus 3,141592 — The *Multiplicand* placed as before.
8347,25 — The *Multiplier* in a reverſe order.

1570796	The *Product* with 5, regard had to 5 times 2
62832	The *Product* with 2, increaſed with 9×2.
21991	*Product* with 7, increaſed with 5×7+9×7.
1257	*Product* with 4, increaſed with 1×4+5×4
94	*Product* with 3, increaſed with 4×3.
25	*Product* with 8, increaſed with 4×8+1×8.
165,6995	The true *Product* as was required.

The Reaſon of this Contraction is very obvious from the whole Operation wrought at large.

Thus 3,141592
52,7438

25	132736	*From hence its evident that all the* Figures *in the Square to the right-hand, are wholly omitted in the former Contraction; And that the Laſt ſingle* Product *here, is the Firſt there; conſequently the Reaſon of placing the* Multiplier *in a Reverſe order muſt needs appear very plain.*
94	24776	
1256	6368	
21991	144	
62831	84	
1570796	0	
165,6995	001296	

Example 2.

Suppoſe it were required to *Multiply* 257,356 with 76,48 and to have only the entire *Product* of *Integers.*

257,356
84,67

18015	
1544	
103	
20	
19682	

The ſame at large. { 257,356 / 76,48 }

20	58848
102	9424
1544	136
18014	92
19682,58688	

The chiefeſt Care and Difficulty that attends theſe Contractions is the true ſetting down of the *Units* place in the *Multiplier* underneath the proper *Figure* of the *Multiplicand*, according to the deſign'd *Product.*

Vi

Viz. In *Example* 1. It was required to have Four places of *Decimal Parts* in the *Product*; therefore the *Units* place of the *Multiplier* was ſet under the Fourth place of *Decimals* in the *Multiplicand*: And in *Example* 2, becauſe it was requir'd to have an entire *Product* of *Integers* only; therefore the *Units* place of the *Multiplier* was ſet under the *Units* place of the *Multiplicand*. This, I ſay, being once Rightly underſtood, will render the Method eaſy in Practice.

Sect. 4. Diviſion of Decimals.

Diviſion is accounted the moſt Difficult part of *Decimal Arithmetick*; In order therefore to make it plain and eaſy, it will be convenient to Reſume what has been ſaid in *Page* 25.

Viz. { *The* Quotient Figure *is always of the ſame Value or Degree with that* Figure *of the* Dividend, *under which the* Units *place of its* Product *ſtands.*

As for Inſtance, Let 294 be *Divided* by 4.

4) 294 (7 { This is not 7 but 70, becauſe the *Units* place of 4×7 ſtands under the *Tens* place of the *Dividend.*
28

14 (3 But this is only 3.
12

Remains (2) Hence $73\frac{2}{4}$ is the *Quotient.*

Now if to the *Remainder* 2 there be Annexed a *Cypher*, thus, 20, and then *Divided* on, it muſt needs follow that the *Units* place of the *Product* ariſing from the *Diviſor* into the *Quotient*, will ſtand under the Annexed *Cypher*; Conſequently the *Quotient Figure* will be of the ſame Value or Degree with the place of that *Cypher*: But that's the next below the *Units* place, therefore the *Quotient Figure* is of the next Degree or Place below *Unity*; That is, in the Firſt place of *Decimal Parts.*

Thus 4) 2,0 (,5

So that 4) 294,0 (73,5 the true *Quotient* required.

This being well underſtood; *Diviſion* of *Decimals* may (in all the various Caſes) be eaſily performed. However, that it may be render'd plain and eaſy, even to the meaneſt Capacity, as poſſible; Let *Diviſion* be again defin'd, as in *Page* 21,

Viz. *If that* Number *which* Divides *another*, *be* Multiplied *with the* Number *which is produced*, *their* Product *will be the* Number Divided.

This Definition alone (if compar'd with the *Rule*, *Page* 61) will afford a General *Rule* for discovering the true Value of the *Quotient Figure* in *Division* of *Decimals*.

Rule { *The places of* Decimal Parts *in the* Divisor *and* Quotient, *being counted together*, *must always be equal in* Number *with those in the* Dividend. *And from this General* Rule *ariseth four Cases.*

Case 1. When the places of *Parts* in the *Divisor* and *Dividend* are Equal, the *Quotient* will be whole *Numbers*.

As in these *Examples*.

8,45) 295,75 (35
2535
4225
4225
(0)

0,0078) ,4368 (56
390
468
468
(0)

Case 2. When the Places of Parts in the *Dividend*, exceed those in the *Divisor*; cut off the Excess for *Decimal Parts* in the *Quotient*. As in these *Examples*.

24,3) 780,516 (32,12
729
515
486
291
243
486
486
(0)

436) 34246,056 (78,546
3052
3726
3488
2380
2180
2005
1744
2616
2616
(0)

,534) ,30438 (,57
2670
3738
3738
(0)

Case 3. When there are not so many Places of Parts in the *Dividend*, as are in the *Divisor*; Annex *Cyphers* to the *Dividend* to make them equal. Then will the *Quotient* be whole *Numbers*, as in *Case* 1.

Examples.

Examples.

Let it be required to *Divide* 192,1 by 7,684, And 441 by ,7875

7,684) 192,100 (25
15368
38420
38420
(0)

,7875) 441,0000 (560
39375
47250
47250
(00)

Case 4. If after *Division* is finished, there are not so many *Figures* in the *Quotient*, as there ought to be Places of Parts by the General *Rule*; supply their defect by prefixing *Cyphers* to it.

Examples.

Let it be required to *Divide* 7,25406 by 957.

957) 7,25406 (,00758 the true *Quotient* required.
6699
5550
4785
7656
7656
(0)

Again ,575) ,0007475 (,0013
575
1725
1725
(0)

Note, When Decimal *Numbers* are to be *Divided* by 10. 100. 1000. 10000. *&c.* that is, when the *Divisor* is an *Unit* with *Cyphers*; *Division* is performed by Removing or Placing the Separating point in the *Dividend*, so many places towards the Left-hand, as there are *Cyphers* in the *Divisor*.

Examples.

10) 5784 (578,4
1000) 5784 (5,784

100) 5784 (57,84
10000) 578,4 (,05784

Note, *These Operations are the direct Converse to those in* p. 62.

I presume it needless to give more *Examples* at Large, only insert a few *Dividends*, and *Divisors*, with their *Quotients*; wherein are contained all the Varieties that can happen in *Division* of *Decimals*.

574) 493066 (859
574) 493,066 (,859
574) 49,3066 (,0859
5,74) 4930,66 (859

5,74) 49,3066 (8,59
5,74) 493066,00 (85900
,0574) 493,0665 (8590
,0574) ,493066 (8,59

There

There is also a compendious way of Contracting *Division*; Like unto that of *Multiplication*, Page 64. by which much Labour may be saved; Especially when the *Divisor* hath many Places of *Decimal Parts* in it: And it's thus performed.

Having determined how many Places of Whole *Numbers* there will be in the *Quotient*, if any at all; or if none, of what value or place the first *Figure* in the *Quotient* will be: Then omit or prick Off one *Figure* of the *Divisor* at each Operation; *viz.* for every *Figure* you place in the *Quotient*, prick Off one in the *Divisor*; having a due regard to the Increase which would arise from the *Figure* so omitted.

Example.

Let it be required to *Divide* 70,23 by 7,9863.

The Work Contracted.

```
7,9863) 70,2300 (8,7938
 ....   63 8904
        -------
         6 3396
         5 5904
         -------
           7492
           7187
           ----
            305
            239
            ---
             66
             64
             --
            (2)
```

The same at Large.

```
7,9863) 70,2300 (8,793
        63 8904
        -------
         6 3396 | 0
         5 5904 | 1
         -------
           7491 | 90
           7187 | 67
           ----- | ---
            304 | 230
            239 | 589
            ---- | ----
             64 | 6410
             63 | 8904
             ---- | ----
              0 | 7506
```

The Work Contracted I presume is so obvious (if compared with the same at Large) that it's needless to give any farther Explanation of it.

Sect. 5. *To* Reduce **Vulgar Fractions** *into* **Decimals**, *and the contrary.*

Any *Vulgar Fraction* being given it may be *Reduced*, or rather Changed into *Decimal Parts* Equivalent to it. Thus,

Rule. { *Annex* Cyphers *to the* Numerator, *and then* Divide *it by the* Denominator, *the* Quotient *will be the* Decimal Parts *Equivalent to the given* Fraction; *or at least so near it as may be thought necessary to approach*

Examp

Example.

'Tis required to *Change* or *Reduce* $\frac{3}{4}$ into *Decimals*.

4) 3,00 (,75 The *Decimal Parts* required.

That is, $\frac{3}{4}=\frac{75}{100}=,75$.

Again $\frac{1}{2}=,5$ Thus 2) 1,0 (,5 And $\frac{1}{4}=,25$ 4) 1,00 (,25

Suppose it were required to *Change* $\frac{4}{7}$ into *Decimals*.

7) 4,0000000000 (,5714285714 &c.$=\frac{4}{7}$.

Note, When the last *Figure* of the *Divisor* (That is, the *Denominator* of the proposed *Fraction*) happens to be one of these *Figures*; viz. 1. 3. 7. or 9. (as in the last *Example*) then the *Decimal Parts* can never be precisely Equal to the given *Fraction*; yet by continuing the *Division* on, you may bring them to be very near the Truth. As in this *Example*; Suppose it was requied to *Change* $\frac{1}{13}$ into *Decimal Parts*.

13) 1,0000 (,076923076923076 &c. *ad infinitum*.

91..

90

78

120

117

30

26

40

39

10

&c.

That is, 0,076923076923076$=\frac{1}{13}$ *fere*.

And from hence it may be farther observed; That in these imperfect *Quotients*, the *Figures* do return again and circulate in the same order as before: As you may easily perceive they begin to do in the seventh place of both these last *Examples*.

As at first.

These being understood, it will be easy to find the *Decimal Parts* Equivalent to any known Part or Parts of *Coin*, *Weights*, *Measures*, or *Time*, &c. If you First *Reduce* the given *Parts* of *Coin*, &c. into a *Vulgar Fraction*, whose *Denominator* is the *Number* of those known *Parts* contained in the *Integer*, and the given *Parts* its *Numerator*.

Examples in Coin, &c.

1. Let it be Required to find the *Decimals* of 16 *s.* 6 *d.* First 16 *s.*$=\frac{16}{20}$ of one *Pound*, and 6 *d.*$=\frac{1}{40}$ of 1 *l.*

But $\frac{16}{20}+\frac{1}{40}=\frac{33}{40}$. Then 40) 33,000 (,825 the *Decimal Parts Required*: That is, ,825=16 *s.* 6 *d.*

Again, Suppose it were *Required* to find the *Decimals* Equal to 3 *l.* 13 *s.* 4 *d.*

Here

Here 3 *l.* is 3 *Integers*, and 13 *s.*=$\frac{13}{20}$ of 1 *l.* and 4 *d.*=$\frac{4}{240}$. But $\frac{13}{20}+\frac{4}{240}=\frac{160}{240}$. Then 240) 160,000 (3,666666, &c. Hence 3 *l.* 13 *s.* 4 *d.*=3,666666, &c. As was required.

2. What are the *Decimals* equal to $7\frac{3}{4}$ Inches, one Foot being made the *Integer*.

First, 7 Inches are $\frac{7}{12}$ of 1 Foot, and $\frac{3}{4}$ of 1 Inch are $\frac{3}{48}$. But $\frac{7}{12}+\frac{3}{48}=\frac{31}{48}$. Then 48) 31,000 (,64583 &c.=$7\frac{3}{4}$ Inches.

3. Let it be Required to Change 8 ℥. 19 *Pwt.* 8 *Grains* into *Decimals*; one ℔ *Troy* being the *Integer*.

These being *Reduced* into their least *Terms*, and *Added* together will become $\frac{4304}{5760}$ of 1 ℔.

Then 5760) 4304,000 (,74722, &c. The *Decimals required*.

And thus may any proposed *Parts* of *Coin*, *Weights*, *Measures*, &c. be *Reduced* or Changed into *Decimal Parts*; which perhaps may at first seem somewhat tedious in Practice, but being a little acquainted with them it will be found very easy; and the ingenious Practitioner will (with a little Consideration) soon find how to *Reduce* them almost mentally; or with the help of a very few *Figures*; without the Use of such large *Tables* as are usually inserted in Books of *Decimal Arithmetick*, or at most they may be contracted into such as these following; which if duly applied to those *Tables* in *Chap.* 3. will be found very useful.

Decimal Tables.

In English Coin.
0,05.....=1 *s.*
0,00416667=1 *d.*
0,00104167=1 *Farthing.*
1, *l.* being the *Integer.*

Troy Weight.
0,05.....=1 *Pwt.*
0,00208333=1 *Grain.*
1, ℥. being the *Integer.*

Apothecaries Weights.
0,125.....=1 *Drachm*
0,04166667=1 ℈
0,00208333=1 *Grain.*
1, ℥. being the *Integer.*

Averdupois Weight.
0,0625....=1 *Ounce.*
0,00390625=1 *Drachm*
1, ℔ being the *Integer.*

Averdupois Great Weight.
0,25.....=$\frac{1}{4}$ *C.*
0,00892857=1 ℔.
0,00055803=1 *Ounce.*
1, *C.* being the *Integer.*

Time.
0,04166667=1 *Hour.*
0,00069444=1 *Minute.*
0,00001157=1 *Second.*
1, *Day* or 24 *Hours* being made the *Integer.*

The Use of these *Tables* will be evident by the following.

Example.

Example.

Let it be Requir'd to find the *Decimal Parts* Equivalent to 7 *s.* 9 *d.* 2 *Farthings.*

irſt, 0,05=1 *s.* Therefore 17×,05=,85 ....=17 *s.*

nd ,004166=1 *d.* Therefore ,004166×9=,037494=9 *d.*

Alſo 2) ,004166 (=,002083=$\frac{1}{2}$ *d.*

Conſequently their *Sum*, viz. 0,889577=17*s.* 9$\frac{1}{2}$*d.*

Now to find the *Value* of *Decimals*, in known *Parts* of *Coin Weights*, &c. is only the Converſe of the former Work. And thus performed.

Multiply *the given* Decimals *with the* Denominator *of the* ulgar Fraction required: *That is*, Multiply *the* Decimals *ith ſuch a* Number *of* Units *as are contained in the next* ower Denomination *of that* Kind *or* Species *which your* ecimal *is of*: *And the* Product *will be the* Number required.

Example.

1. What is the *Value* of 0,825 *Decimals* of 1 *Pound Sterling.* hat is, how many *Shillings*, *Pence*, &c.=,825. Firſt, the next ower *Denomination* is 20, becauſe 20 *s.* make one *Pound.*

Therefore 0,825

20

Shillings 16,500 And *Parts* of 1 *Shilling.*

12

Pence 6,000 Anſwer 0,825=16 *s.* 6 *d.*

Again, What are the known *Parts* of *Engliſh Coin* Equal to 666666 *Decimals.*

Here the 3 *Integers* are 3 *Pounds.* Then ,666666

20

Shillings 13,333320

12

nſwer 3,666666=3 *l.* 13 *s.* 4 *d.* 666640

33332

Pence 3,999840

What is the *Value* of 0,74722 Parts of 1 ℔ *Troy.*

rſt, ,74722 Then, ,96664 Again, ,33280

12 20 24

1 49444 *Pwts.* 19,33280 1 3312

7 4722 6 656

8,96664 ℥. *Pwt.* *Gr.* 7,98720

Theſe Collected are 8. 19. 8. very near. And

And thus any propoſed *Number* of *Decimals* may be turn'd or chang'd into the known *Parts* of what they repreſent, *viz.* Whether they be *Parts* of *Coin*, *Weights*, *Meaſures*, or *Time*, &c.

I have omitted inſerting more *Examples* of this kind, becauſe I take the Excellency, and indeed the chief Uſe of *Decimal Fractions* to conſiſt more in *Geometrical Computations*, than in the Common or Practical *Parts* of *Arithmetick* (as will appear further on) although even in thoſe they are very uſeful upon ſeveral Accounts; Eſpecially in the *Computations* of *Intereſt* and *Annuities*, &c. (But of that more in its proper Place.) I ſhall therefore conclude this *Chapter* with a Remark or Two upon the *Nature* and *Properties* of *Fractions* in General.

If any given *Number* (whether it be *Whole* or *Mix'd*) be *Multiplied* with a *Fraction* either *Vulgar* or *Decimal*, the *Product* will be Leſs than the *Multiplicand*, in ſuch a Proportion as the *Multiplying Fraction* is Leſs than an *Unit* or 1.

That is, as the Denominator *of the* Fraction *is to its* Numerator; *ſo will the given* Number *be to the* Product.

Therefore, whenever any *Number* is to be *Multiplied* with a *Fraction*, whoſe *Numerator* is an *Unit*. *Divide* that *Number* by the *Denominator* of the *Fraction*, and the *Quotient* will be the *Product required*. Thus $12 \times \frac{1}{4} = 3$. And $12 \div 4 = 3$. Again, $12 \times \frac{1}{2} = 6$. And $12 \div 2 = 6$, *&c.*

From hence it follows, that if any *Number* be *Divided* by a *Fraction*, the *Quotient* will be greater than the *Dividend*, by ſuch a Proportion as *Unity* is greater than the *Dividing Fraction*. Thus $12 \div \frac{1}{4} = 48$, *viz.* $\frac{1}{4} : 1 :: 12 : 48$, *&c.* But the Truth of theſe will be beſt underſtood after the next *Chapter*.

CHAP. VI.

Of Continued Propoſitions, *and how to* Change *or Vary the Order of Things.*

Sect. 1. *Concerning* Arithmetical Progreſſion, *uſually called* Arithmetical Proportion Continued.

WHEN any *Rank* or *Series* of *Numbers* do either Increaſe or Decreaſe by an Equal *Interval* or Common *Difference*, thoſe *Numbers* are ſaid to be in *Arithmetical Progreſſion*.

As { 1 . 2 . 3 . 4 . 5 . 6 . 7 . &c. } { Here the *Interval* or
{ 7 . 6 . 5 . 4 . 3 . 2 . 1 . } { Common *Difference* is 1.
Or { 2 . 4 . 6 . 8 . 10 . 12 . 14 . &c. } { Here the Common
{ 1 . 3 . 5 . 7 . 9 . 11 . 13 . &c. } { *Difference* is 2.

And ſo of any other *Series*, whoſe Common *Difference* is . 4 . 5 . &c.

Lemma 1.

If any three *Numbers* be in *Arithmetical Progreſſion*; the *um* of the two *Extreams* (*viz.* the Firſt and Laſt) will be qual to the Double of the *Mean* or middle *Number*.

As in theſe 2 . 4 . 6 . Or 3 . 6 . 9 . Or 3 . 7 . 11.
Viz. 2+6=4+4. Or 3+9=6+6. And 3+11=7+7. &c.

Lemma 2.

If any Four *Numbers* are in *Arithmetick Progreſſion*, the *Sum* of e two *Extreams* will be Equal to the *Sum* of the Two *Means*.

As in theſe 2 . 4 . 6 . 8. Or 3 . 6 . 9 . 12.
Viz. 2+8=4+6. And 3+12=6+9. &c.

Corollary 1.

From theſe two Lemma's *it's eaſy to conceive, that if never many* Numbers *be in* Arithmetick Progreſſion, *the* Sum *of the wo* Extreams *will be Equal to the* Sum *of any Two* Means, *at are Equally diſtant from thoſe* Extreams.

As in theſe, 2 . 4 . 6 . 8 . 10 . 12 . 14 . 16.
Then 2+16=4+14=6+12=8+10.

Or if the *Number* of *Terms* be odd as theſe.

2 . 4 . 6 . 8 . 10 . 12 . 14 . 16 . 18. &c.
Then 2+18=4+16=6+14=8+12=10+10.

Lemma 3.

Every *Series* of *Numbers* in *Arithmetick Progreſſion* is ompoſed of the *Interval* or Common *Difference*, ſo often peated as there are *Terms* in the *Progreſſion* except the Firſt.

As in theſe, 1. 3. 5. 7. 9. 11. 13. 15. 17. &c.
Here the *Interval* or Common *Difference* being Two, it will 1+2=3. 3+2=5. 5+2=7. 7+2=9. 9+2=11. +2=13. 13+2=15. 15+2=17. &c.

Corollary 2.

Hence it's evident, that the Difference *betwixt the Two* xtreams (*viz.* 1 *and* 17) *is compoſed of the Common* Difference, ultiplied *into the* Number *of all the* Terms *excepting the irſt.*

As in the aforeſaid *Progreſſion*, 1. 3. 5. 7. 9. 11. 13. 15. 17.

The *Number* of *Terms* without the *First* is 8
The Common *Difference* is 2 } *Multiply*

The *Difference* betwixt the two *Extreams* 16

Proposition 1.

In any *Series* of *Numbers* in *Arithmetick Progression*, the Two *Extreams*, and the *Number* of *Terms* being given; thence to find the *Sum* of all the *Series*.

Theorem. *Multiply the Sum of the two Extreams into the Number of all the Terms; and Divide the Product by* 2. *The Quotient will be the Sum of all that Series. Per Corol.* 1.

Example 1.

'Tis *Required* to find the *Number* of all the Strokes a Clock strikes in one whole Revolution of the Index, *viz.* Twelve Hours.

Here 1+12=13 the *Sum* of the two *Extreams*.
12 the *Number* of all the *Terms*.

26
13

Then 2) 156 (78. The *Number* of Strokes *Required*.

Example 2.

Suppose one *Hundred* Eggs were placed in a Right Line a Yard distant from one another; and the first Egg were a Yard from a Basket; whether may a Man gather up those 100 Eggs singly one after another, still returning with every Egg to the Basket and put it in, before another Man can Run Four Miles. That is, which will run the greater *Number* of Yards.

In this Question 200+2=202 Is the *Sum* of the Two *Extr.*
And 100 Is the *Number* of all the *Terms*.

Then 2) 20200 (10100 The *Number* of Yards he runs that takes up the Eggs.

Now 4 Miles=7040 Yards } The Yards he runs that takes up the
But 10100—7040=3060 } Eggs more than the other.

Proposition 2.

In any *Series* of *Numbers* in *Arithmetick Progression*, the Two *Extreams* and *Number* of *Terms* being given; thence to find the Common Difference of all the *Terms* in that *Series*.

Theorem 2. *The Difference betwixt the Two Extreams being Divided by the Number of Terms less one Unit or* 1. *The Quotient will be the Common Difference of the Series. Per Corol.* 2.

Example

Example 1.

One had Twelve Children that Differed alike in all their Ages; the Youngeſt was Nine Years old, the Eldeſt was Thirty Six and a half; what was the Difference of their Ages, and the Age of each?

Here 36,5—9=27,5 The *Difference* of the Two *Extreams*.
And 12—1=11. The *Number* of *Terms* leſs an *Unit*.
Then 11) 27,5 (2,5. The Common Difference *Required*.
Conſequently 9+2,5=11,5 The Age of the Youngeſt but one.
And 11,5+2,5=14 The Age of the Youngeſt but two. And ſo on for the reſt. *per Corol.* 2.

Example 2.

A Debt is to be Diſcharged at Eleven ſeveral Payments to be made in *Arithmetick Progreſſion*. The Firſt Payment to be Twelve *Pounds Ten Shillings*, and the Laſt to be *Sixty Three Pounds*. What's the whole Debt, and what muſt each Payment be?

Per Theorem 1. Find the whole Debt thus:

12,5+63=75,5 The *Sum* of the *Extreams*.
11 The *Number* of *Terms*.

755
755

2) 830,5 (415,25=415 *l.* 5 *s.* The whole Debt.

Then *per Theorem* 2. Find the Common *Difference* of each Payment.

Thus 63—12,5=50,5 The *Difference* of the *Extreams*.
And 11—1=10 The *Number* of *Terms* leſs 1.
Then 10) 50,5 (5,05=5 *l.* 1 *s.* The Common *Difference*.

l. *s.* *l.* *s.* *l.* *s.*
Conſequently 12 . 11+5 . 1=17 . 11 The Second Payment.
l. *s.* *l.* *s.* *l.* *s.*
And 17 . 11+5 . 1=22 . 12 The Third Payment, &c.

Example 3.

A Man is to Travel from *London* to a certain Place in Ten Days, and to go but Two Miles the firſt Day, encreaſing every Days Journey by an equal Exceſs; ſo that the laſt Days Journey may be Twenty Nine Miles; What will each Days Journey be, and how many Miles is the Place he goes to, diſtant from *London*?

First 29—2=27 The *Difference* of the *Extreams*.
And 10—1=9 The *Number* of *Terms* leſs 1.
Then 9) 27 (3. The Common *Difference*.
Conſequently 2+3=5. The Second Days Journey.
And 5+3=8. The Third Days Journey, &c.
Again 29+2=31 The *Sum* of the *Extreams*.
10 The *Number* of *Terms*.
2) 310 (155 The Diſtance *Required*.

There are Eighteen *Theorems* more relating to Queſtions in *Arithmetick Progreſſion*; but becauſe they would *Require* a great many Words to ſhew the Reaſon of them: I therefore refer the Reader to the Second Part, *viz.* That of *Algebra*, where he may find their *Analitical Inveſtigation*.

Sect. 2. Concerning Geometrical Proportion *Continued*; ſometimes called *Geometrical Progreſſion*.

When a *Rank* or *Series* of *Numbers* do either Increaſe by one Common *Multiplicator*, or Decreaſe by one common *Diviſor*; thoſe *Numbers* are ſaid to be in *Geometrical Proportion* continu'd.

As { 2 . 4 . 8 . 16 . 32 . &c. here 2 is the common *Multiplier*
{ 64 . 32 . 16 . 8 . 4 . &c. here 2 is the common *Diviſor*
Or { 2 . 6 . 18 . 54 . 162 . &c. here 3 is the common *Multiplier*
{ 162 . 54 . 18 . 6 . 2 . here 3 is the common *Diviſor*.

Note, The common *Multiplier* (or *Diviſor*) is called the *Ratio*; and it ſhews the Habitude or Relation the *Numbers* have to one another, *viz.* whether they are Double, Triple, Quadruple, &c. which *Euclid* thus defines.

Ratio (*or* Rate) *is the mutual Habitude or Reſpect of Two* Magnitudes (*conſequently Two* Numbers) *of the ſame kind each to other, according to Quantity*. Eu. 5. Def. 3.

Proportion (rather *Proportionality*) is a Similitude of *Ratio's*. Eu. 5. Def. 4.

So that there cannot be Leſs than Three Terms to Form a *Proportionality* or *Similitude* of *Ratio's*; and if but Three Terms, the ſecond muſt ſupply the Place of Two, As in theſe 2 . 4 . 8 . That is, 2 : 4 : : 4 : 8 . (of : : ſee *page* 5.)

Here 4 the *Middle Term* ſupplies the place of Two *Terms*, to wit, of the Second and Third; 8 bearing the ſame Reaſon,

Likeneſs,

Likeneſs, or *Proportion* to 4, As 4 doth to 2. *Viz.* As 2 : is to 1 :: So is 4 : to 8.

Lemma 1.

If three *Numbers* are proportional, the *Rectangle* or *Product* of the Two Extreams; *viz.* of the Firſt and Laſt Terms will be equal to the *Square* of the Mean or Middle Term. (20 *Eucl.* 7.)

As in theſe 2 : 4 : : 4 : 8 Here 8×2=16 the *Product* of the Extreams.

And 4×4=16 the *Square* of the Mean. *Ergo* 8×2=4×4.

Corol. 1.

Hence it follows, That if the *Product* of any Two *Numbers* be Equal to the *Square* of a Third *Number*; thoſe Three *Numbers* will be in Proportion.

Lemma 2.

If four *Numbers* are proportional, the *Product* of the Two Extreams, will be equal to the *Product* of the Two Means. (19 *Euclid* 7.)

As in theſe, 2 : 4 : : 8 : 16. Here 16×2=32.

And 8×4=32. Conſequently 16×2=8×4.

Corol. 2.

From hence it follows, That if the Product *of any Two* Numbers, *be Equal to the* Product *of any other Two* Numbers, *thoſe Four* Numbers *are Proportionals.*

And from theſe two *Lemma*'s it will be eaſy to conceive, that never ſo many *Numbers* are in continued Proportion; the *Product* of the two *Extreams*, will be equal to the *Product* of any Two *Means*, that are equally diſtant from the *Extreams.*

As in theſe 2 . 4 . 8 . 16 . 32 . 64 . *&c.*

Here 64×2=32×4=16×8. *&c.* And if the *Number* of Terms be Odd,

As in theſe 2 . 4 . 8 . 16 . 32 . 64 . 128 *&c.*

Then 128×2=64×4=32×8=16×16.

Note, *The Character made uſe of to ſignify continued Proportionals is* ∴

In

In every *Series* of ∺ (viz. *of continued Proportionals*) tha *Number* which is compared to another, is called the *Antecedent* of the *Ratio*; and that *Number* to which it is compared, i called its *Conſequent*.

As in theſe, 2 : 4 : : 4 : 8. Here 2 is the *Antecedent*, an 4 is the *Conſequent*; and 4 the middle Term is an *Antecedent* t 8 its *Conſequent*: Whence it follows, that in every *Series* of ∺ a the middle Terms between the Firſt and Laſt are both *Antece dents* and *Conſequents*.

As in theſe, 2 . 4 . 8 . 16 . 32 . 64. &c. Here 4 . 8 . 16 32 are both *Conſequents* and *Antecedents*.

For 2 : 4 : : 4 : 8 : : 8 : 16 : : 16 : 32 : : 32 : 64 &c.

So that all the Terms except the laſt are *Antecedents*. An all the Terms except the firſt are *Conſequents*.

Lemma 3.

If never ſo many *Numbers* are Proportional, it will be: A any one of the *Antecedents* is to its *Conſequent*: So will the *Sum* of all the *Antecedents* be; To the *Sum* of all the *Conſequents* (12 *Euclid* 5.)

That is, in the foregoing *Series*.

2 : 4 : : 2+4+8+16+32 : 4+8+16+32+64.

For it's evident, that 4+8+16+32+64 the *Sum* of all th *Conſequents*, is double to 2+4+8+16+32 the *Sum* of all th *Antecedents*; As 4 is to 2, according to the *Ratio*, and woul have been Triple, or Quadruple, &c. had the *Ratio* been 3 4, &c.

Note, *In every* Series *of* ∺ the *Ratio* is found by *Dividi* any of the *Conſequents* by its *Antecedent*.

As in theſe, 2 : 6 : : 6 : 18 : : 18 : 54 : : 54 : 162. Here 2) 6 (3 the *Ratio*. Or 6) 18 (3 &c.

From the Second and Third *Lemma*'s may be raiſed Tw General *Theorems* or *Rules*, for finding the *Sum* of any *Seri* in ∺ without a continued *Addition* of all the *Terms*.

Let the *Series* 2 . 4 . 8 . 16 . 32 . 64 . 128 . be given find its *Sum*.

Suppoſe z = the *Sum* of all the *Terms*.
Then will $z-128$ = the *Sum* of all the *Antecedents*.
And $z-2$ = the *Sum* of all the *Conſequents*.
But 2 : 4 : : $z-128$: $z-2$. *per Lemma* 3.
Ergo $4z-512=2z-4$. *per Lemma* 2.

Cor

Conſequently $4z-2z=512-4$.

Theorem. $z=\frac{512-4}{4-2}$ In words at Length thus.

heorem 1. *From the Product of the Second and Laſt Terms, Subſtract the Square of the Firſt Term; that Remainder being Divided by the Second Term Leſs the Firſt will give the Sum of all the Series.*

Or the Firſt Term, the common *Ratio*, and the Laſt Term only given. Then,

heorem 2. *Multiply the Laſt Term into the Ratio, and from their Product Subſtract the Firſt Term; Divide that Remainder by the Ratio Leſs Unity or* 1, *and it will give the Sum of all the Series.*

For $4z-2z=512-4$. As above.
onſequently $2z-z=256-2$ *viz.* the laſt Divided by 2.

Then $z=\frac{256-2}{2-1}$ *Theorem* 2.

Example. Let 2 . 6 . 18 . 54 . 162 . 486. be the given *Series*. ere 2 is the Firſt Term, 3 is the *Ratio*, and 486 the Laſt *Term*.

But $486\times3=1458$. And $1458-2=1456$.
Then $3-1=2)\ 1456\ (728$ the Sum required.
That is, $728=2+6+18+54+162+486$.

In either of theſe *Theorems* it is required to have the Laſt *rm* known (the which in a Long *Series* of ÷ will be very ious to come at by a continued *Multiplication*, &c. It will refore be convenient to ſhew how to obtain either the Laſt *rm* or any other *Term*, whoſe place is aſſigned; without pro- cing all the *Terms*.

n order to that, it will be neceſſary to premiſe the Coherence Similitude that is betwixt *Numbers* in *Arithmetical Progreſſion* d thoſe in *Geometrical Proportion*.

f to any *Series* of *Numbers* in ÷ when the firſt *Term* is not *Unit* or 1, there be Aſſigned a *Series* of *Numbers* in *Arith- tical Progreſſion*, beginning with an *Unit* or 1, and whoſe mmon *Difference* is 1. Called *Indices* or *Exponents*.

Thus, 1 . 2 . 3 . 4 . 5 . 6 . 7 *Indices*.
2 . 4 . 8 . 16 . 32 . 64 . 128 &c. ÷

Then

Then will the *Addition* or *Subſtraction* of any Two of thoſe *Indices* (or *Numbers* in *Arithmetick Progreſſion*) directly correſpond with the *Product*, or *Quotient* of their reſpective *Terms* in the *Series* of ÷

That is, { As 3+4=7 / So 8×16=128 the *Seventh Term* in ÷

Again, { As 6+4=10. / So 64×16=1024. the *Tenth Term* in ÷

Or, { As 7—3=8. / So 128÷8=16. } Or, { As 6—2=4 / So 64÷4=16 } &c.

But if the *Series* of ÷ begin with an *Unit*, the *Indices* muſt begin with a *Cypher*.

As in theſe, { 0 . 1 . 2 . 3 . 4 . 5 . 6. &c. / 1 . 2 . 4 . 8 . 16 . 32 . 64.

Now by the help of theſe *Indices*, and a few of the Firſt *Terms* in any *Series* of ÷ It is plain that any *Term* whoſe *Place* or *Diſtance* from the Firſt *Term* is Aſſigned, may be ſpeedily obtained without producing the Whole *Series*.

Example 1.

A Man bought a Horſe, and was to give a *Farthing* for the First Nail, Two for the Second, Four for the Third, &c. In ÷ the *Number* of Nails was to be 7 in every Shoe, *viz.* 28 Nails in all. What muſt he have paid for the Horſe?

First { 0 . 1 . 2 . 3 . 4 . 5. *Indices*. / 1 . 2 . 4 . 8 . 16 . 32. *Farthings* in ÷

Then, { 5+5=10 / 32×32=1024 And { 10+10=20 / 1024×1024=1048576

Again, { 4+3=7 / 16×8=128 Laſtly, { 20 + 7 = 27 / 1048576×128=134217

Which is here to be accounted the 28 and laſt *Term*. Becauſe the Firſt Term in the *Series* is 1. which doth neither *Multiply* nor *Divide*.

Now this 134217728 being the *Number* of *Farthings* to be paid for the Laſt Nail, by it the common *Ratio* which is 2, and the Firſt *Term* which is 1. may be found the *Sum* of all the *Series*, *per Theorem* 2.

134

134217728
2

268435456 From this *Product Subſtract* 1.
iz. 268435456—1=268435455. Then 2—1=1 the *Diviſor.*
Conſequently 268435455 is the *Sum* of all the *Series* or *Price* the Horſe in *Farthings*; which being brought into *Pounds*, c. (See *Page* 46) will be 279620 *l.* 5 *s.* 3 *d.* 3 *qrs.*

Example 2.

A cunning Servant agreed with a Maſter (unskill'd in *Numbers*) ſerve him Eleven Years without any other Reward for his rvice but the *Produce* of one *Wheat Corn* for the Firſt Year; nd that *Product* to be Sow'd the Second Year; and ſo on from ear to Year until the End of the Time. Allowing the Encreaſe be but in a Ten-fold *Proportion.*
'Tis *Required* to find the Sum of the whole *Produce.*

rſt { 1 . 2 . 3 . 4 . 5 . *Indices* or *Years.*
10 . 100 . 1000 . 10000 . 100000 *Wheat Corns* in ÷

hen { As 4 + 2 = 6
So 10000×100=1000000. the 6th Years *Produce.*

nd { 6 + 5 = 11
1000000×100000=100000000000. The Eleventh or ſt Years *Produce.*

Then, (either by *Theorem* 1 or 2) the *Sum* of all the *Series* ll be, 111111111110 Corns. Now it may be Computed from ge 31 and 34, that 7680 Wheat Corns round and dry out of e middle of the Ear, will fill a Statute Pint. If ſo;
Then 7680) 111111111110 (14467592 Pints, but 64 Pints e contained in a Buſhel.
Therefore 64) 14467592 (226056$\frac{1}{8}$ Buſhels. Suppoſe it to ſold for 3 Shillings the Buſhel;

Then { 226056$\frac{1}{8}$
3

Shillings 678168$\frac{3}{8}$=33908 *l.* 8 *s.* 4$\frac{1}{2}$ *d.* A very good ecompence for Eleven Years Service.

There are ſeveral pretty Queſtions Reſolved by *Numbers* in *rithmetical Progreſſion*; And by thoſe in ÷ which the ingeniS Learner will eaſily perceive hereafter; *viz.* When we come the *Solution* of *Queſtions* relating to *Intereſt* and *Annuities*, &c.

There is alſo a Third Kind of *Proportion*, called *Muſical* which being but of little or no common uſe, I ſhall therefo give but a ſhort account of it.

Muſical Proportion or *Habitude* is, when of Three *Numbe* the Firſt hath the ſame *Proportion* to the Third; As th *Difference* between the Firſt and Second hath to the *Differen* between the Second and Third.

As in theſe, 6 . 8 . 12. *viz.* 6 : 12 : : 8—6 : 12—8

If there are Four *Numbers* in *Muſical Proportion*, the Fi will have the ſame *Proportion* to the Fourth; As the *Differen* between the Firſt and Second hath to the *Difference* betwe the Third and Fourth.

As in theſe 8 . 14 . 21 . 84.

Here 8 : 84 : : 14—8=6 : 84—21=63.

That is, 8 : 84 : : 6 : 63.

The Method of finding out *Numbers* in *Muſical Proporti* is beſt expreſſed by Letters; as ſhall be ſhewed in the *Algebr ick Part.*

Sect. 3. *How to* Change *or* Vary *the Order of Things*, &c.

This being a Thing not Treated of in any common Books *Arithmetick* (that I have had the opportunity of peruſing) ma me think it would be acceptable to the young Learner to kn how Oft its poſſible to Vary or Change the *Order* or *Poſition* any Propoſed *Number* of *Things*.

As how many ſeveral Changes may be Rung upon a Propoſed *Number* of Bells; Or how many ſeveral Variati may be made of any Determined *Number* of *Letters*; Or a other things expoſed to be Varied.

The Method *of finding out the* Number *of* Changes, *is by continual* Multiplication *of all the* Terms *in a* Series *of* Ari metical Progreſſionals; *whoſe Firſt* Term, *and Common* Dif rence *is* Unity *or* 1. *And Laſt* Term *the* Number *of Thi propoſed to be Varied*, viz. 1×2×3×4×5×6×7, *&c.* As w appear from what follows.

1. If the things propoſed to be varied are only Two, th admit of a double *Poſition*, as to order of Place; And no mo

Thus, $\left\{\begin{matrix} 1 & . & 2 \\ 2 & . & 1 \end{matrix}\right\}=2=1\times 2.$

2. And if Three things are propoſed to be varied, they m

changed Six ſeveral ways; As to their order of Places; And more.

For beginning with 1 there will be { 1 . 2 . 3 / 1 . 3 . 2

Next beginning with 2 there will be { 2 . 1 . 3 / 2 . 3 . 1

Again, beginning with 3 and it will be { 3 . 1 . 2 / 3 . 2 . 1

Which in all make 6 or 3 times 2. *viz.* 1×2×3=6

3. Suppoſe Four things are Propoſed to be varied; hen they will admit of 24 ſeveral Changes, as to their Order Different Places.

For beginning the Order with 1 it will be { 1 . 2 . 3 . 4 / 1 . 2 . 4 . 3 / 1 . 3 . 2 . 4 / 1 . 3 . 4 . 2 / 1 . 4 . 2 . 3 / 1 . 4 . 3 . 2

Here is Six Different Changes.

And for the ſame Reaſon there will be 6 Different Changes hen 2 begins the Order, and as many when 3 and 4 begins the rder; which in all is 24=1×2×3×4. And by this Method of roceeding, it may be made evident, that 5 Things admit of 120 veral Variations or Changes; and 6 Things of 720, &c. As this following *Table.*

The Number *of Things propoſed to be varied.*	*The manner how their ſeveral Variations are Produced.*	*The different Changes or Variations every one of the propoſed* Numbers *can admit of.*
1	1×1	=1
2	1×2	=2
3	2×3	=6
4	6×4	=24
5	24×5	=120
6	120×6	=720
7	720×7	=5040
8	5040×8	=40320
9	40320×9	=362880
10	362880×10	=3628800
11	3628800×11	=39916800
12	39916800×12	=479001600

Theſe may be thus Continued on to any Aſſigned *Number*. Suppoſe to 24 the *Number* of Letters in the Alphabet, which will admit of 620448401733239439360000 ſeveral Variations.

From theſe Computations may be ſtarted ſeveral pretty, and indeed very ſtrange Queſtions.

Examples.

Six Gentlemen that were Travelling met together by Chance at a certain Inn upon the Road, where they were ſo pleaſed with their Hoſt, and each others Company, that in a Frolick they made a Contract to ſtay at that Place, ſo long as they together with their Hoſt, could ſit every Day in a Different Order or Poſition at Dinner; which by the foregoing Computation will be found near 14 Years. For they being made 7 with their Hoſt, will admit of 5040 Different *Poſitions*; but 5040 being *Divided* by $365\frac{1}{4}$ the *Number* of Days in one Year, will give 13 Years and 291 Days. A very pretty Frolick indeed.

I have been told, (That before the great Fire of *London*, which happened *Anno* 666) there was 12 Bells in St. *Mary le Bows* Church in *Cheapſide*, *London*. Suppoſe it were Required to tell how many ſeveral Changes might have been Rung upon thoſe 12 Bells: and at a moderate Computation how long all thoſe Changes would have been Ringing but once over.

Firſt, 1×2×3×4×5×6×7×8×9×10×11×12=479001600 the *Number* of *Changes*.

Then ſuppoſing there might be *Rung* 10 *Changes* in one *Minute*: *viz.* 12×10=120 *Strokes* in a *Minute*, which is 2 *Strokes* in a *Second* of *Time*; now according to that rate there muſt be allowed 47900160 *Minutes* to *Ring* them once over in all their Different *Changes*; viz. 10) 479001600 (47900160.

In one *Year* there is 365 *Days*, 5 *Hours*, and 49 *Minutes*; which being *Reduced* into *Minutes*, is 525949.

Then 525949) 47900160 (91 *Years* and 26 *Days*.

So long would thoſe 12 *Bells* have been continually *Ringing* without any Intermiſſion, before all their Different *Changes* could have been truly *Rung* but once over. 'Tis ſtrange, and ſeems almoſt incredible, that a few Things ſhould produce ſuch Varieties.

But that which ſeems yet more ſtrange and ſurpriſing, yea, en impoſſible to thoſe who are not a little vers'd in the power *Numbers*, is, that if two *Bells* more had been Added to the oreſaid 12, they would have advanc'd the *Number* of *Changes* nd conſequently the Time) beyond common belief. For 14 *ells* would require (at the ſame rate of *Ringing* as before) a- ut 16575 *Years* to *Ring* all their Different *Changes* but once er.

And if it were poſſible to *Ring* 24 *Bells* in *Changes*, and at the me Rate of 10 *Changes* in a *Minute*, which is 2 *Strokes* in one *econd*; they would require more than 11700000000000000 *ars* to *Ring* them but once over in all their Different *Changes*; may eaſily be computed from the precedent *Table*.

CHAP. VII.

Of Proportion Disjunct; *commonly called the* Golden Rule.

Roportion Disjunct, or the **Golden Rule**, is either *Direct* or *Reciprocal*, called *Inverſe*. And thoſe are both *Simple* and *mpound*.

Sect. 1.

Direct Proportion is, when of Four *Numbers*, the Firſt bear- h the ſame *Ratio* or *Proportion* to the Second; As the Third th to the Fourth.

As in theſe 2 : 8 :: 6 : 24.

Conſequently, the greater the Second *Term* is, in reſpect to e Firſt; the greater will the Fourth *Term* be, in reſpect to e Third.

That is, as 8 the Second *Term*, is 4 times greater than 2 the rſt *Term*: So is 24 the Fourth *Term*, 4 times greater than 6 e Third *Term*.

Whence it follows, that if Four *Numbers* are in *Direct Pro- rtion*, the *Product* of the Two *Extreams* will always be Equal the *Product* of the Two *Means*, as well in *Disjunct* as in ntinued *Proportion*; according to *Lemma* 2. *page* 77.

For As 2 : 2 × 4 :: 6 : 6 × 4. Or As 3 : 3 × 5 :: 6 : 6 × 5.
But 2 × 6 × 4 = 2 × 4 × 6. Or 3 × 6 × 5 = 3 × 5 × 6.

hat is, the *Product* of the *Extreams* is equal to that of the *eans*.

Again,

Again, the Leſs the ſecond *Term* is, in reſpect to the Firſt, the Leſs will the Fourth *Term* be, in reſpect to the Third.

As in theſe 18 : 6 :: 12 : 4.

That is, 18 : 18 ÷ 3 :: 12 : 12 ÷ 3.

But 18 × 12 ÷ 3 = 18 ÷ 3 × 12. *Viz.* 18 × 4 = 6 × 12.

Conſequently 2 . 8 . 6 . 24. And 18 . 6 . 12 . 4 . are true *Proportionals*, *per Corol.* 2. *page* 77.

From theſe Conſiderations, comes the Invention of finding a Fourth *Number* in *Proportion* to any Three given *Numbers*. *Whence it's called the Rule of Three.*

For if the Second *Number Multiplied* into the Third, be equal to the Firſt *Multiplied* into the Fourth, it is eaſy to conceive that if the *Product* of the Second and Third be *Divided* by the Firſt, the *Quotient* muſts needs be the Fourth *Number*. For that *Number* which *Divides* another, be *Multiplied* into the *Quotient* produced by that *Diviſion* ; their *Product* will be Equal to the *Number Divided*. See *Page* 21.

As in theſe 2 : 8 :: 6 : 24. Here 8 × 6 = 48 = 24 × 2.

But if 24 × 2 = 48. then will 48 ÷ 2 = 24. Or 48 ÷ 24 = 2.

Note, Any *Four Numbers* in *Direct Proportion* may be varied ſeveral ways. As in theſe.

Viz. If 2 : 8 :: 6 : 24. Then 2 : 6 :: 8 : 24.
And 6 : 24 :: 2 : 8. Or 24 : 6 :: 8 : 2. &c.

Theſe Variations being well underſtood, will be of no ſmall uſe in the true ſtating of any Queſtion in this Rule of Three.

When Three *Numbers* are given, and it is required to find a Fourth *Proportional* ; the greateſt Difficulty (if there be any) will be in the Right ſtating the *Queſtion*, or *Abſtracting* the *Numbers* out of the Words in the *Queſtion*, and placing them down in their proper Order.

Now this will be very eaſy, if it be truly conſidered, that always Two of the Three given *Terms*, are only ſuppoſed, and aſſign or limit the *Ratio* or *Proportion*. The Third moves the *Queſtion* ; And the Fourth gives the *Anſwer*.

As for Inſtance ; If 3 *Yards* of *Cloth* coſt 9 *Shillings* : What will 6 *Yards* coſt at the ſame *Rate* or *Proportion* ?

Here 3 *Yards*, and 9 *Shillings*, are Two ſuppoſed *Numbers* that imply the Rate ; as appears by the Word [if], *viz.* if 3 *Yards* coſt 9 *Shillings* (then comes the *Queſtion*) What will 6 *Yards* coſt ?

N. B

N. B. The *Term* which moves the *Queſtion* hath generally ne of theſe Words before it; *viz.* **What will? How many? ow long? How far?** or, **How much?** &c.

Then (carefully obſerve this; *viz.*) The Firſt *Term* in the *uppoſition* muſt always be of the ſame kind and *Denomination* ith that *Term* which moves the *Queſtion*. And the *Term* ught will always be of the ſame kind and *Denomination* with e Second *Term* in the Suppoſition.

Thus, { *yds ſhil. yds ſhil.* 3 : 9 :: 6 : —— } Then

All *Queſtions* in *Direct Proportion* may be *Anſwered* by hree ſeveral *Theorems*.

heorem 1. { *Multiply the Second and Third Terms together, and Divide their Product by the Firſt Term; the Quotient will be the Anſwer required.*

yds ſhil. yds ſhil.
Thus 3 : 9 :: 6 : 18. The Anſwer.

6

3) 54 (18 *Shillings*, { becauſe the Second *Term* was *Shillings*.

heorem 2. { *Divide the Second Term by the Firſt, then Multiply the Quotient into the Third Term; and their Product will be the Anſwer required.*

yds ſhil. yds ſhil.
3 : 9 :: 6 : 18.
Thus 3) 9 (=3. Then 3×6=18, as before.

heorem 3. { *Divide the Third Term by the Firſt, then multiply the Quotient into the Second Term, and their Product will be the Anſwer.*

yds ſhil. yds ſhil.
3 : 9 :: 6 : 18.
Thus) 6 (=2. And 9×2=18. as before.

Here you ſee that all the Three *Theorems* are Equally true; the Firſt is moſt General, and uſually practiſed. Yet the o laſt may be readily performed when either the Second or ird *Term* can be *Divided* by the Firſt; And will be found of gular Uſe in the *Rules of Fellowſhip*, &c. as will appear fur- r on.

Queſt.

Queſt. 2. If 8 Pounds of *Tobacco* coſt 14 *Shillings*; What w
half a hundred Weight (*viz.* 56 Pounds) coſt at the ſame Rat

Thus 8 ℔ : 14 *s.* : : 56 ℔ : 4 *l.* 18 *s.* The *Anſwer.*

14

224
56

8) 784 (=98 *s.*=4 *l.* 18 *s.*

Or thus 8) 56 (=7. Then 14×7=98 *s.* as before.

Queſt. 3. If 14 *Shillings* will buy 8 Pounds of *Tobacco*; H
much will 4 *l.* 18 *s.* buy after the ſame Rate?

Stated thus, 14 *s.* : 8 ℔ : : 4 *l.* 18 *s.*=98 *s.* : ——
Then 98×8=784. And 14) 784 (56 ℔ The *Anſwer.*

Queſt. 4. If half a hundred Weight of *Tobacco* be worth
18 *s.* How much may I buy for 14 *Shillings* at that Rate?

Stated thus, 4 *l.* 18 *s.*=98 *s.* : 56 ℔ : : 14 *s.* : ——
Then 56×14=784. And 98) 784 (8 ℔ The *Anſwer.*

Queſt. 5. Suppoſe 4 *l.* 18 *s.* will buy 56 Pounds of *Tobacc*
What will 8 Pounds of the ſame *Tobacco* coſt?

This *Queſtion* is thus Stated, 56 ℔ : 4 *l.* 18 *s.*=98 *s.* : : 8 ℔ :-
Then 98×8=784. And 56) 784 (=14 *s.* The *Anſwer.*

Note, The Three laſt *Queſtions* are only the Second vari
being propoſed purely to give an Inſtance how any *Queſtion*
this *Rule of Three* may be varied, according to *Page* 86.

Queſt. 6. What will three quarters of a *Yard* of *Velvet* c
when the Price of 21 *Yards* and a half is worth 22 *l.* 10 *s.* 6
This *Queſtion* truly Stated will ſtand

Thus, 21 $\frac{1}{2}$ *yds* : 22 *l.* 10 *s.* 6 *d.* : : $\frac{3}{4}$ To the *Anſwer.*

Which may be found three ſeveral ways; *viz.* by *Reductio*
by *Vulgar Fractions*; and by *Decimals.*

1. By *Reduction.* Bring the Firſt and Third *Terms* into
Denomination; *viz.* into *Quarters*, and *Reduce* the Seco
Term into its Leaſt *Denomination*, *per Sect.* 4. *Page* 42.

Thus 21 $\frac{1}{2}$=86 *Quarters.* And 22 *l.* 10 *s.* 6 *d.*=5406 *Pence*
Then 86 : 5406 : : 3 : 15 *s.* 8 $\frac{20}{86}$ *d.* For 5406×3=16218.

And 86) 162 18 (=188$\frac{50}{86}$ *d.* Then 188$\frac{50}{86}$ *Pence*=15 *s.* *l.* 2$\frac{14}{43}$ *Farthings*; the *Answer* required.

2. The same *Question* stated in *Vulgar Fractions* will stand
us 21$\frac{1}{2}$=$\frac{43}{2}$: 22$\frac{21}{40}$=$\frac{901}{40}$ :: $\frac{3}{4}$: (See *Sect.* 3. *page* 50.)
en $\frac{901}{40}\times\frac{3}{4}=\frac{2703}{160}$. And $\frac{43}{2}$) $\frac{2703}{160}$ (=$\frac{5406}{6880}$ *page* 55, 56.
These $\frac{5406}{6880}$ Parts of a *Pound* are brought into *Shillings* by *ultiplying* the *Numerator* with 20, and *Dividing* the *Product* its *Denominator*, &c.
Thus 5406×20=108120. And 6880) 108120 (15 *s.*
And there *Remains* 4920. Again 4920×12=59040.
Then 6880) 59040 (8 *d.* and $\frac{50}{86}$ &c. as before.

3. The same wrought by *Decimal Fractions* will be thus;
1$\frac{1}{2}$=21,5 22 *l.* 10 *s.* 6 *d.*=22,525 and $\frac{3}{4}$=0,75
Therefore 21,5 : 22,525 :: 0,75 : to the *Answer.*
Then 22,525×0,75=16,89375
And 21,5) 16,89375 (07857 *l.*=15 *s.* 8 *d.* 2 *far.* $\frac{272}{1000}$.

Quest. 7. If 2 *C.* 3 *qrs.* 21 ℔ of *Sugar* cost 6 *l.* 15 *s.* 8 *d.* What will 12 *C.* 2 *qrs.* cost at the same Rate?
That is 2 *C.* 3 *qr.* 21 ℔ : 6 *l.* 15 *s.* 8 *d.* :: 12 *C.* 2 *qr.* To what?

4	20	4
11 *qrs.*	121 *s.*	50 *qrs.*
28	12	28
88	250	1500 ℔
22	121	

viz. 308+21=329 ℔ : 1460 *d.* :: 1400 ℔ : ——
Then 1460×1400=2044000. And 329) 2044000 (6212$\frac{3}{4}$ *d.* *l.* 17 *s.* 8$\frac{3}{4}$ *d.* the *Answer* required.

The same *Question* stated in *Decimals* will stand
Thus 2,9375 : 6 0833 :: 12,5 : To the *Answer.*
Then 6,0833×12,5=76,04125 which being *Divided* by 75 will give 25,8863 &c. the *Answer* in *Decimals*, which ght into *Coin*, will be 25 *l.* 17 *s.* 8$\frac{3}{4}$ *d.* as before.
ote, *When the First* Term *is an* Unit *or* 1, *the* Question *is* vered *by* Multiplication *only.*

xample. Suppose I give 5 *Shillings* 4 *Pence* for one *Ounce* *lver*, What must I pay for 32$\frac{1}{2}$ *Ounces* at the same Rate?
That is 1$\frac{2}{3}$: 5 *s.* 4 *d.* :: 32$\frac{1}{2}\frac{2}{3}$: To, &c.
hich is best Stated thus 1 : 64 *d.* :: 32 5 :

Then 32,5×64=2080 *d.*=8 *l.* 13 *s.* 4 *d.* the *Answer* required. For 1 neither *Multiplies* nor *Divides*.

When the Second or Third *Term* is an *Unit* or 1, then the *Question* is *Answered* by *Division* only. As in this *Example*.

If a *Silver Tankard* weighing 21 *Ounces*, coſt 5 *l.* 19 *s.* What is that an *Ounce*?

Thus 21 *oz.* : 5 *l.* 19 *s.*=1195 *s.* :: 1 : 5 *s.* 8 *d.* To the *Answer*.
That is 21) 119 (=5 *s.* $\frac{14}{21}$=5 *s.* 8 *d.*

The Proof of all *Questions* in the *Rule of Three Direct*, may be eaſily conceiv'd from what hath been already ſaid; viz. That the *Product* of the Firſt and Fourth *Terms*, muſt always be Equal to the *Product* of the Second and Third *Terms*.

Or otherwiſe, by varying the *Queſtion*, as in the Second, Third, Fourth, and Fifth *Queſtions*.

I ſhall conclude this *Section* with inſerting a few *Queſtions* and their *Anſwers*; leaving their Work for the Learner's Practice.

Queſt. 1. What will the *Carriage* of 17 *C.* 3 *qrs.* 11 ℔ come to, at the Rate of 7*s.* the *Hundred*?

Anſwer 6 *l.* 4 *s.* 11 $\frac{1}{4}$ *d.*

Queſt. 2. If 6 *l.* 4 *s.* 11 $\frac{1}{4}$ *d.* be paid for the *Carriage* of 17 *C.* 3 *qrs.* 11 ℔; What was paid for the Carriage of 1 ℔?

Anſwer 3 *Farthings*.

Queſt. 3. A *Grocer* bought 3 *C.* 1 *qr.* 14 ℔ *weight* of *Cloves* at the Rate of 2 *s.* 4 *d.* *per Pound*, and ſold them for 52 *l.* 14 *s.* Whether did he gain or loſe by the Bargain, and *how much*?

Anſwer he gained 8 *l.* 12 *s.*

Queſt. 4. A *Draper* bought of a *Merchant* Eight Packs of *Cloth*; every Pack had Four Parcels in it; And each Parcel contained Ten Pieces; Every Piece was Twenty-ſix *Yards*; He gave after the Rate of four *Pounds* ſixteen *Shillings* for 6 *Yards*. What came the Eight Packs to, and what was it worth *per Yard*?

Anſw. They came to 6656 *l.* And is worth 16 *s.* *per Yard*.

Queſt. 5. A *Merchant* bought 436 *Yards* of Broad *Cloth* at 8 *s.* 6 *d.* *per Yard*; And ſold it again for 10 *s.* 4 *d.* *per Yard*. What did he gain by the 436 *Yards*?

Anſw. he gain'd 39 *l.* 19 *s.* 4 *d.*

Qu

Queſt. 6. A *Goldſmith* bought a Wedge of *Gold*, which *weighed* 4 lb 3 *oz.* 8 *pw.* for 514 *l.* 4 *s.* What did he pay *per Ounce*?

Anſw. 3 *l. per Ounce.*

Queſt. 7. What will 48 *oz.* 17 *pw.* 20 *Grains of Silver* Plate come to, at the Rate of 5 *s.* 6 *d. per Ounce*?

Anſw. 13 *l.* 8 *s.* 10 $\frac{3}{4}$ *d.*

Queſt. 8. If in Four *Weeks* one ſpend 13 *s.* 4 *d.* How long will 3 *l.* 6 *s.* laſt at that Rate?

Anſw. 6 *Years*, 47 *Days*, 2 *Hours*, 24.

Queſt. 9. What will the one eighth part of a *Ship* be worth, when the half is valued at 1015 *l.* 10 *s.*?

Anſw. 253 *l.* 17 *s.* 6 *d.*

Queſt. 10. The *Sun* is ſaid to perform one entire Revolution, (or 360 Degrees) in the Space of 365 Days, 5 Hours, 48 Minutes, and 57 Seconds of Time, called a *Tropical* or *Solar Year*; How much doth it Move in one Day?

Anſw. 59 . 8 . 19 &c.

Queſt. 11. If $\frac{5}{8}$ of a *Yard* of *Velvet* coſt $\frac{2}{3}$ of a *Pound Sterling*, what will $\frac{1}{16}$ of a *Yard* coſt of the ſame *Velvet* at that Rate?

Anſw. $\frac{16}{240}$ = 1 *s.* 4 *d.*

Queſt. 12. Suppoſe 2 *l.* and $\frac{3}{8}$ of $\frac{1}{3}$ of a *Pound Sterling* will buy 3 *Yards* and $\frac{2}{3}$ of $\frac{3}{5}$ of a *Yard of Cloth*, How much will $\frac{3}{4}$ of a *Yard* coſt at that Rate?

Anſw. $\frac{2295}{4896}$ of a *Pound* = 9 *s.* 4$\frac{1}{2}$ *d.*

Sect 2. *Of* Reciprocal Proportion; *uſually called* The Rule of Three Inverſe.

Reciprocal Proportion is, when of Four *Numbers* the Third (*viz.* that which moves the *Queſtion*) beareth the ſame *Ratio* to the Firſt; As the Second does to the Fourth.

Therefore, the Leſs the Third *Term* is, in reſpect to the Firſt; the Greater will the Fourth *Term* be, in reſpect to the Second.

Example 1.

If Sixteen Men can do a Piece of Work in Six Days; How many Days muſt Eight Men require to do the ſame Work, at the ſame Rate of Working?

Here 'tis plain that Eight Men muſt needs have more time than 16 Men to do the ſame Work. Conſequently the

greater the third *Term* is, in reſpect to the Firſt, the leſſer wi
the fourth *Term* be, in reſpect to the Second.

Example 2. If 8 Men can do a Piece of Work in 12 Day
How many Days will 16 Men require to do the ſame Work
Here it is plain the fourth *Term* muſt be leſs than the ſecon
becauſe 16 Men undoubtedly can do the ſame Work in leſs Tim
than 8 Men can.

From theſe Conſiderations, compared with thoſe in Page 8
'twill be eaſy to perceive, whether the Terms of any propoſ
Queſtion *are* Direct *or* Reciprocal Proportion.

For when according to the true Meaning or Deſign of a
Queſtion *in* Proportion. *More* requires *More*, or *Leſs* requir
Leſs, *the* Terms *are in* Direct Proportion, *as in the laſt* Section.

But if More require *Leſs*, or *Leſs* require *More*, *(as abov*
then the Terms *will be in* Reciprocal Proportion.

The Manner of placing down the *Propoſed Terms* is the ſam
in both *Rules*, *viz.* The firſt *Term* in the *Suppoſition* muſt be
the ſame Kind and *Denomination* with the third *Term* whic
moves the *Queſtion*; and the *Term* ſought muſt be of the ſam
Kind and *Denomination* with the ſecond *Term* in the Suppoſitio
As in the two laſt *Examples*.

		Men	*Days*		*Men*	*Days*
Thus, in	*Example* 1.	16 :	6	::	8 :	—
	Example 2.	8 :	12	::	16 :	—

The Queſtion being truly Stated, obſerve this *Theorem*.

Theorem. *Multiply the firſt and ſecond Terms togeth*
and Divide their Product by the third Term, t
Quotient will be the Anſwer Required.

Thus in the ſecond *Example* 12×8=96.

Then 16) 96 (=6 Days the Anſwer *Required*.

That is, 16 Men may do the ſame Work in 6 Days as 8 M
can do in 12 Days.

Now the Reaſon of this *Operation*, (and conſequently of t
Theorem) is grounded upon this Conſideration; *viz.* If 8 M
require 12 Days to do the Work, 'tis plain that one Man wou
require 8 times 12 Days=96 Days to do the ſame Work, b
if one Man can do it in 96 Days, moſt certain 16 Men can do
in one 16th part of that Time. Therefore 96 *Divided* by 16 w
give the Anſwer *Required*, *viz.* 16) 96 (6 as before, *&c.*

Queſt. 3. Suppoſe 800 Soldiers were Beſieged in a Town, an
their Victuals were computed to ſerve them Two Months (or
Days) How many of thoſe Soldiers muſt depart the Gariſon, th
the ſame Victuals may ſerve the remaining Soldiers 5 Months.

The Queſtion truly Stated will ſtand

Month Soldier Month Soldier

Thus, 2 : 800 : 5 : ———

2

———

5) 1600 (320 So many Soldiers may ſtay in the Gariſon.

Conſequently, 800—320=480 Soldiers that muſt go out of Gariſon, which is the *Anſwer* required.

Queſtion 4. *A.* Borrowed of his Friend *B.* 250 *l.* for Six nths promiſing to do him the like Kindneſs upon Demand: me time after *B.* deſires *A.* to Lend him 400 *l.* the Queſtion how long *B.* may keep the 400 *l.* to be fully ſatisfied for his ner Kindneſs to *A.*

Anſw. 3 Months 21 Days.

Thus, 250 *l.* : 6 Months :: 400 *l.* : ———

6

———

400) 1500 (3 Months.

— —

12

———

3

28 Days in one Month.

———

4) 84 (21 Days.

Queſtion 5. If a Penny White Loaf ought to weigh Eight nces *Troy Weight*, when Wheat is ſold for Six Shillings Six nce the Buſhel, What muſt it weigh when Wheat is ſold for r Shillings the Buſhel?

Anſw. 10 *oz* . 18 *pw* . 8 *gr.*

us 6 *s.* 6 *d.*=78 *d.* : 8 *oz.* :: 4 *s.*=48 *d.* To the *Anſwer.*

8

———

48) 624 (13 *oz.* the *Anſwer* required.

48

———

144

144

———

(0)

The Proof of this *Inverſe Rule* is eaſily deduced from its erations; *viz.* The *Product* of the firſt and ſecond *Terms*, t be equal to the *Product* of the third and fourth *Terms*.

Note,

Note, Any Queſtion that falls under this *Inverſe Rule of Reciprocal Proportion*, may be ſo Stated as to have its *Terms* in *Direct Proportion*; by only changing the Places of the firſt and third *Terms* in the *Queſtion*. Thus,

Queſtion 6. If a Field will feed Eighteen Horſes for Seven Weeks, How long will it feed Forty Two Horſes at the ſame Rate of Feeding?

First, 18 Horſes : 7 Weeks : : 42 Horſes : 3 Weeks.

Here the *Terms* are ſtated *Inverſely*, as before.

Otherwiſe thus, 42 Horſes : 7 Weeks : : 18 Horſes : 3 Weeks. Then $18 \times 7 = 126$. And $126 \div 42 = 3$ Weeks. The *Anſwer* required.

Sect. 3. *Of* Compound Proportion; *commonly called* The Double Rule of Three.

Compound Proportion (as 'tis here meant) is, when there are Five Numbers given to find out a Sixth Proportional; and this is generally performed by a *Double Poſition*; that is, by Stating and Working the Queſtion at Two *Operations*, either in Direct or Reciprocal *Proportion*, according as the Queſtion requires.

And therefore it's called, The Double Golden Rule; *or* Double Rule of Three.

The *Double Rule* Direct is, when the Sixth *Term* or *Number* ſought, is found by Two *Operations*, both of them in *Direct Proportion*.

Example 1. If a Hundred Pounds gain Six Pounds Intereſt in Twelve Months, How much will Three Hundred Pounds gain in Nine Months, at the ſame Rate?

First 100 *l.* : 6 *l.* : : 300 *l.* : 18 *l.*

6

100) 1800 (18 *l.* { The Intereſt of 300 for Twelve Months.

Months *Months*

Then, 12 : 18 *l.* : : 9 : 13 *l.* 10 *s.*

9

12) 162 (13 *l.* 10 *s.* The *Anſwer* required.

I ſuppoſe the Learner will eaſily conceive the Reaſon of these Two *Operations*. For, First it's plain by Direct *Proportion*, that if 100 *l.* gain 6 *l.* in Twelve Months, 300 *l.* will gain 18 *l.* in the ſame Time, and at the ſame Rate.

And by the ſame *Rule* 'tis plain; that if 12 Months will oduce or give 18 *l.* Intereſt for 300 *l.* then 9 Months muſt needs ve 13 ½ for the ſame *Sum*, *viz.* 300 *l.*

The *Double Rule of Three Inverſe* is, when the Sixth *Term Number* ſought is found at Two *Operations* (as before). But e of them Requires an Anſwer in *Reciprocal Proportion.*

Queſtion 2. If 6 Buſhels of Oats will ſerve 4 Horſes 8 Days, ow many Days will 21 Buſhels ſerve 16 Horſes, at the ſame ate of Feeding?

This Queſtion being parted into Two *Poſitions*, the Firſt will thus:

If 6 Buſhels of Oats will ſerve 4 Horſes 8 Days, How many ays will 21 Buſhels ſerve them?

Here 'tis plain, that 21 Buſhels will ſerve them Longer than 6 uſhels; therefore the Firſt *Poſition* falls in *Direct Proportion.*

Buſh. Days Buſh. Days
Thus, 6 : 8 :: 21 : 28.

8

6) 168 (28 Days

That is, if 6 Buſhels will ſerve 4 Horſes 8 Days, 21 Buſhels ill ſerve them 28 Days.

The next *Poſition* muſt be to find how Long the ſaid 21 Buſhels ll ſerve 16 Horſes at the ſame Rate of Feeding: 'Tis plain, t 21 Buſhels cannot ſerve 16 Horſes ſo many Days as they l ſerve 4 Horſes; therefore this ſecond *Poſition* falls in *Reci-ocal Proportion.*

Horſes Days Horſes Days
Thus, 4 : 28 :: 16 : 7 the *Anſwer* Required.

After the like manner any Queſtion in the *Double Rule of ree* may be Anſwered by two ſingle *Poſitions*, if Care be taken Stating them Right, *viz.* Whether their *Operation* muſt be formed by the ſingle *Rule Direct*, or *Inverſe.*

But all Queſtions in this *Double Rule*, where five *Numbers* e propoſed to find a Sixth, may more eaſily and readily be ſwered by one *General Theorem*; which compriſeth both the *rect* and *Inverſe Rules*; without conſidering either of them, ng Deduced from the ſingle *Operations* before-going.

But firſt you muſt carefully Note, That in all *Queſtions* of s *Nature* Three of the Five propoſed *Terms* are always Conditional

Conditional and Suppoſed; And that the other Two moves th Queſtion. As for Inſtance in *Example* 1.

Viz. If 100 *l.* will gain 6 *l.* in 12 Months: Theſe Thr Terms are only Suppoſed or Conditional. Then comes th Queſtion; What will 300 *l.* gain in 9 Months? Now, in orde to raiſe the *General Theorem*, let us ſuppoſe, inſtead of *Numbe* theſe Letters.

Viz. Let $\begin{cases} P=100. & \text{The } \textit{Principal}. \\ T=12. & \text{The } \textit{Time}. \\ G=6. & \text{The } \textit{Gain}. \end{cases}$ } { In the Suppoſition of any propoſed Queſtion.

And, $\begin{cases} p=300. & \text{The } \textit{Principal}. \\ t=9. & \text{The } \textit{Time}. \\ g=13,5 & \text{The } \textit{Gain}. \end{cases}$ } { The Three *Terms* wherein the Queſtion lies.

Then $P : G :: p : \frac{Gp}{P} =$ { The *Product* of the Two *Means* D *vided* by the firſt *Extream*.

That is, $100 : 6 :: 300 : \frac{300 \times 6}{100} = 18.$ } { Which is the Firſt part of the Queſtion.

Then $T : \frac{Gp}{P} :: t : g$ } { Which is the Second part of the Queſtion.

Viz. 12 : 18 :: 9 : 13,5

Ergo $Tg = \frac{Gpt}{P}$ } { That is, the *Product* of the *Extreams* Equal to that of the *Means*.

Conſequently, $TgP = Gpt$. Is the *Theorem*.

This *Theorem* affords Two *Rules* by which all *Queſtions* i this *Double Rule of Three*, or rather of Five *Numbers*, may b Reſolved; due regard being had to the true placing down of th propoſed *Terms*, which muſt be thus:

Always place the Three *Conditional Terms* in this Order; L that *Number* which is the Principal Cauſe of *Gain*, *Loſs*, *Action*, &c. *(viz. P.)* be put in the Firſt Place; That *Numb* which denotes the Space of *Time*, or *Diſtance* of Place, & *(viz. T.)* be put in the Second Place. And that *Number* whic is the *Gain*, *Loſs*, or *Action*, &c. *(viz. G.)* be put in the Thir Place. Now according to theſe Directions, the *Condition Terms* of the laſt Queſtion will ſtand thus; *P . T . G.*

That done, place the other Two *Terms* which move the Que ſtion, underneath thoſe of the ſame Name,

Thus $\begin{cases} P \,.\, T \,.\, G. \\ p \,.\, t \,. \end{cases}$

The

Then if the *Blank* or *Term* ſought, fall under the Third Place, n this Queſtion,

It will be $\left\{ \frac{Gpt}{TP} = g. \right.$ Which gives this *Rule.*

le 1. Multiply *the Three laſt* Terms *together for a* Dividend ; *and the Two firſt together for a* Diviſor ; *the* Quotient *ariſing from them will be the Sixth* Term.

That is, in our Propoſed *Example* 1.

Thus $6 \times 300 \times 9 = 16200$ The *Dividend.*

And $100 \times 12 = 1200$ The *Diviſor.*

Then 1200) 16200 ($13\frac{1}{2}$ The *Anſwer.* As before.

But if the *Blank* or *Term* ſought fall under the Firſt Place, then

It will be $\left\{ \frac{TgP}{tG} = p \right.$

Or if the *Blank* fall under the Second Place,

It will be $\left\{ \frac{TgP}{Gp} = t \right.$ Either of theſe give this *Rule.*

le 2. Multiply *the Firſt, Second and Laſt* Terms *together for a* Dividend : *And the other Two together for a* Diviſor ; *the* Quotient *ariſing from them will be the Sixth* Term.

nd becauſe our *Example* 2. falls under the Conſideration of Direct and *Reciprocal Proportion*, let it be here propos'd n.

iz. If 6 Buſhels of Oats will ſerve 4 Horſes 8 Days ; How y Days will 21 Buſhels ſerve 16 Horſes, *&c.*

the *Terms* of this *Queſtion* be placed down as before Direct- hey will ſtand

hus *Horſes Days Buſhels*
4 . 8 . 6 *Terms* in the *Suppoſition.*
16 . 21

ere the *Blank* falls under the Second *Place*, therefore it muſt ound by the Second *Rule.*

hus $4 \times 8 \times 21 = 672$ the *Dividend.*

nd $16 \times 6 = 96$ the *Diviſor.*

Then 96) 672 (7 the *Anſwer* as before.

Queſt. 3. What *Principal* or *Stock* will *Gain* 20*l.* in 8 *Mont* at 6 *per Cent. per Annum*?

Prin. Time Gain
100 . 12 . 6 *Terms* in the *Suppoſition.*
. 8 . 20

In this *Queſtion* the Blank falls under the firſt *Place*, there fore it muſt be found by the Second *Rule.*

Thus 100×12×20=24000 the *Dividend.*
And 8×6=48 the *Diviſor.*
Then 48) 24000 (500*l.* the *Anſwer* required.

The Proof of all *Queſtions* in this Double *Rule* of Five *Numbers*, is beſt perform'd by varying the *Queſtion*; viz. by ſtating it in another Order, as in the laſt *Example*: Thus,

If 100*l.* *Gain* 6*l.* in 12 *Months*, what will 500*l.* *Gain* in *Months*?

The *Anſwer* to this *Queſtion* muſt be 20*l.* if the Work of the laſt *Example* be True.

Prin. Time Gain.
Stated thus { 100 . 12 . 6 } then *per Rule* 1.
{ 500 . 8 . }

500×8×6=24000. And 100×12=1200.
Then 1200) 24000 (20*l.* the *Anſwer*, &c.

Queſt. 4. If Two Men can do 12 Rods of Ditching in 6 Days *How many* Rods may be done by 8 Men in 24 Days, at the ſame Rate of working?

Anſw. 192 Rods.

Queſt. 5. If the Carriage of 5 *C.* 3 *qrs. Weight*, 150 *Miles* coſt 3*l.* 7*s.* 4*d.* What muſt be paid for the Carriage of 7 *C.* 2 *qrs.* 25 lb. *Weight*, 64 *Miles*, at the ſame Rate?

Anſw. 1*l.* 18*s.* 7½*d.*

Queſt. 6. If 8 Men deſerve 2*l.* Wages for 5 Days Work, *How much* will 32 Men deſerve for 24 Days, at the ſame Rate?

Anſw. 38*l.* 8*s.*

Queſt. 7. Suppoſe a *Hundred Pounds* would defray the Expences of Five Men for Twenty-two Weeks and Six Days, *How long* would Twelve Men be in ſpending of one *Hundred* and *Fifty Pounds*, and at the ſame Rate?

Anſw. 14 Weeks and 2 Days.

CHAP

CHAP. VIII.

f Trading in Company, uſually call'd the Rule of Fellowſhip; *alſo* Bartering, *and* Exchanging *of Coins*, &c.

THE *Rule of Fellowſhip* is that by which the Accompts of ſeveral Partners Trading in a Company, are ſo adjuſted or ade up, that every Partner may have his juſt part of the *Gain*, ſuſtain his juſt part of the *Loſs*; according to his proportion or are of Money he hath in the Joint-Stock: Now this falls under wo Conſiderations, called the Single and Double *Rules of llowſhip*.

Sect. 1. *The* Single Rule of Fellowſhip; viz. *That without* Time.

By the *Single Rule of Fellowſhip* is adjuſted the Accompts of oſe Partners that put all their ſeveral and perhaps different ms of Money, into a common Stock at one and the ſame me; and therefore it's uſually called the *Rule of Fellowſhip* thout Time: Now all *Queſtions* of this Nature are Anſwered ſo many ſeveral *Operations* in the *Rule of Three* Direct, as re are Partners in the Stock.

For; *As the Total Sum of Money in the Stock is in Proportion the whole Gain or Loſs*; *So is every Man's particular part that Stock*; *To his particular ſhare of that Gain, or Loſs.*

Queſt. 1. There are Three Partners, Suppoſe *A*, *B*, and *C*, ke a Joint-Stock of 96 *l.* in this manner.

A, puts in 24 *l.* *B*, puts in 32 *l.* and *C*, puts in 40 *l.* with s 96 *l.* they Trade and Gain 12 *l.* 'Tis required to find each n's true Part of that Gain.

The firſt Operation for *A*'s Part of the Gain will ſtand

Thus 96 *l* : 12 *l* : : 24 *l* : 3 *l* = *A*'s Gain.

Secondly 96 *l* : 12 *l* : : 32 *l* : 4 *l* = *B*'s Part of the Gain.

Again 96 *l* : 12 *l* : : 40 *l* : 5 *l* = *C*'s Part of the Gain.

Proof 3 *l* + 4 *l* + 5 *l* = 12 *l* the whole Gain.

That is, if the Sum of each Man's particular Gain, Amount the whole Gain, the Work is true; *if not, ſome Error is mitted which muſt be found out.*

Note. Theſe Operations will be very much abbreviated, if work them by *Theorem* 2. *Page* 87. For here 96 is a common *ecedent*, and 12 is the Common Conſequent in all the Three portions.

Therefore 96 : 12 :: 1 : 0,125 is a common *Multiplicator*.

Then 24×0,125=3 *l.* for *A*'s Part
And 32×0,125=4 *l.* for *B*'s Part } As before.
Again 40×0,125=5 *l.* for *C*'s Part

Now this Method is more readily perform'd than the other Especially when the Partners are many; becauſe one *Single Diviſion* ſerves for all the Work.

Queſt. 2. Three *Merchants*, *A*, *B*, and *C*, Freight a Ship with 248 *Tun* of *Wine*: thus, *A* Loaded 98 *Tun*, *B* 86 *Tun* and *C* 64 *Tun*. By Extremity of Weather the Seamen were forced to caſt or throw 93 *Tun* of it over-board. *How much* of this Loſs muſt each *Merchant* ſuſtain?

First 248 : 93 :: 1 : 0,375 the common *Multiplier*.
Then 98×0,375=36,75 for *A*'s Loſs.
And 86×0,375=32,25 for *B*'s Loſs.
Again 64×0,375=24, for *C*'s Loſs.

Proof 93,00=the whole Loſs.

Now if the *Queſtion* were to find *how much* of the Remaining Wine that was ſaved, belongs to *A*, to *B*, and to *C*.

Then 98—36,75=61,25 belongs to *A*.
And 86—32,25=53,75 belongs to *B*.
And 64—24, =40, belongs to *C*.

That is, *A* ought to have 61 *Tun* and 63 *Gallons*. *B* ought to have 53 *Tun* and 189 *Gallons*. And *C* ought to have 40 *Tun* of what was Left.

Queſt. 3. Suppoſe ſix Men, *viz.* *A*, *B*, *C*, *D*, *E*, and *F*, make a Joint-Stock of 2558 *l.*

		l.	*s.*	*Decimals.*
Thus	*A* puts in	654	. 10	= 654,5
	B ———	543	. 15	= 543,75
	C ———	480	. 00	= 480,
	D ———	254	. 10	= 254,5
	E ———	365	. 05	= 365,25
	F ———	260	. 00	= 260,

The whole Stock 2558 . 00=2558,00 according to the *Queſtion*.

With this Stock of 2558 *l.* they Trade Eighteen *Months*, and in 831 *l.* 7 *s.* 'Tis required to find every Man's Part or Share of that Gain.

Note, *Altho' the Time of Trading*, viz. *Eighteen Months be mentioned in the* Queſtion, *yet it's no way concerned in anſwering of it; as you may obſerve by the following Work.*

First, 2558 *l.* : 831,35 *l.* :: 1 *l.* : 0,325 *Decimal Parts.*
Conſequently, 1 *l.* : 0,325 :: 654,5 : 212,7125. That is,

A's Stock	654,5	×0,325=212,7125	for *A*.
B's Stock	543,75	×0,325=176,71875	for *B*.
C's Stock	480,	×0,325=156,000	for *C*.
D's Stock	254,5	×0,325= 82,7125	for *D*.
E's Stock	365,25	×0,325=118,70625	for *E*.
F's Stock	260,	×0,325= 84,5	for *F*.

That is,

	l. *parts.*	*l.*	*s.*	*d.*
A Gains	212,7125	=212	. 14	. 03
B ——	176,71875	=176	. 14	. 04 $\frac{1}{2}$
C ——	156,0000	=156	. 00	. 00
D ——	82,7125	= 82	. 14	. 03
E ——	118,70625	=118	. 14	. 01 $\frac{1}{2}$
F ——	84,5	= 84	. 10	. 00
Proof. *Sum*	831,35	=831	. 07	. 00 *the Gain.*

I have omitted reſolving this Queſtion according to the uſual thod (as before directed) of finding every Man's particular t of the Gain by the *Golden Rule*, as in the Firſt Work of ample 1. Leaving that for the Learner's Practice.

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t. 2. *The* **Double Rule of Fellowſhip**; *or that with* Time.

This is uſually called the *Double Rule of Fellowſhip*, becauſe ry particular Man's Money is to be conſidered with relation to time of its Continuance in the Joint-Stock.

Queſtion 1. *A* and *B* join in Partnerſhip upon theſe Terms, . *A* agrees to lay down 100 *l.* and to imploy it in *Trade* 3 *nths*: Then *B* is to lay down his 100 *l.* and with the whole ck of 200 *l.* they are to Trade 3 *Months* more. Now at the d of that Time, they find their whole Gain to to be 21 *l.* s required to know what each Man's part of the Gain ought to according to his Stock, and the time of imploying it.

Here

Here it is but reaſonable to conclude, that *A.* ought to Gain more than *B.* notwithſtanding their Stocks of *Money* are Equal; becauſe *A.* imploy'd his Money a Longer *Time* than *B.*

Now for Solving of this *Queſtion*, Let us ſuppoſe *A's* 100 *l.* imployed the firſt 3 *Months* to Gain Z = a Sum as yet unknown; then it muſt Gain 2Z in 6 *Months*; and to find what *B.* muſt Gain it will be,

l. *Months.*
100 . 6 . 2Z = *A's* Gain
100 . 3 . To *B's* Gain } *per Rule* 1. *Page* 97.

$$Ergo\ \frac{100 \times 3 \times 2Z}{100 \times 6} = B\text{'s Gain.}$$

But *A's* Gain Added to *B's* Gain muſt = 21 *l.* the whole Gain by the *Queſtion.*

$$\text{Therefore } 2Z + \frac{100 \times 3 \times 2Z}{100 \times 6} = 21\ l.$$

That is, $100 \times 6 \times 2Z + 100 \times 3 \times 2Z = 21 \times 100 \times 6.$

Which contracted is, $900 \times 2Z = 21 \times 600.$

Conſequently, $2Z = \frac{21 \times 600}{900}$ which gives the following *Analogy.*

Viz. 900 : 21 :: 600 : 2Z = 14 *l.* for *A's* Gain.
And 900 : 21 :: 100 × 3 = 300 : 7 *l.* for *B's* Gain.

Now this way of Arguing hath not only Reſolved the preſent *Queſtion*; but it alſo affords (and *demonſtrates*) a General Rule for Reſolving all *Queſtions* of this *Nature*, be the Partners never ſo many.

Rule. *Multiply every particular Man's Stock, with the Time it is employed; then it will be, As the Sum of all thoſe Products; Is to the whole Gain (or Loſs). So is every one of thoſe Products; To its proportional part of that whole Gain (or Loſs).*

Queſtion 2. Three Merchants *A. B.* and *C.* enter into Partnerſhip thus; *A.* puts into the Stock 65 *l.* for 8 *Months*; *B.* puts in 78 *l.* for 12 *Months*; and *C.* puts in 84 *l.* for 6 *Months.* With theſe they *Traffick*, and Gain 166 *l.* 12 *s.* 'Tis required to find each Man's Share of the Gain, Proportionable to his Stock and Time of employing it.

1. *As*

1. *A*'s Stock 65*l.* × 8 *Months, the time it was employed*=520
2. *B*'s Stock 78*l.* × 12 *Months, the time it was employed*=936
3. *C*'s Stock 84*l.* × 6 *Months, the time it was employed*=504

The *Sum* of thoſe *Products* is, 1960

Then, according to the *Rule*, the ſeveral Proportions will ſtand thus,

1960 : 166,6 :: 520 : 44,2 =44*l.* 4*s.* 0*d.* for *A.*
1960 : 166,6 :: 936 : 79,56=79*l.* 11*s.* 2½*d.* for *B.*
1960 : 166,6 :: 504 : 42,84=42*l.* 16*s.* 9½*d.* for *C.*

The whole Gain=166*l.* 12*s.* 0*d.*

Or you may work as in ſome of the former *Examples, viz.* by finding the proportional Part of the Gain due to one Pound, *&c.*

Thus 1960 : 166,6 :: 1 : 0,085 the common *Multiplier.*
Then 520×0,085=44,2 for *A.*
And 936×0,085=79,56 for *B.* } *&c.* As before.
Alſo 504×0,085=42,84 for *C.*

Queſtion 3. Six Merchants, *viz. A. B. C. D. E.* and *F.* enter into Partnerſhip, and compoſe a Joint-Stock in this manner;

	l.	*s.*		
Viz. *A* puts in	64	10	4½	
B ——	78	15	6	
C ——	100	00	8¼	*Months.*
D ——	80	10	12	
E ——	74	12	9½	
F ——	125	15	7	

for

They *Traffick*, and Gain 258*l.* 18*s.* 4½*d.* 'Tis required to find every Man's Share of the Gain, according to his Stock and *Time* it was employed.

The ſeveral Stocks of Money, and their reſpective Times being Firſt brought into *Decimals*, and then Multiplied together will produce theſe following Products.

	l.	*Months.*	
A's Stock	64,5	×4,5	*the time it was employed*=290,25
B's Stock	78,75	×6,	*the time it was employed*=472,5
C's Stock	100,	×8,25	*the time it was employed*=825,0
D's Stock	80,5	×12,	*the time it was employed*=966,0
E's Stock	74,6	×9,5	*the time it was employed*=708,7
F's Stock	125,75	×7,	*the time it was employed*=880,25

The *Sum* of thoſe *Products*=4142,7

Then

Then if you work by the common Way; it will be 4142,7 : 258,91875 :: 290,25 : 18,140625 = 18 *l.* 2 *s.* 9¾ *d.* for *A*'s part of the Gain; And so on for the rest.

But if you Work by the easiest way, *viz.* by finding the proportional Part of the Gain due to one *Pound*.

Thus 4142,7 : 258,91875 :: 1 : 0,0625

			l.	*s.*	*d.*	
Then	290,25	×0,0625=18,140625	=18 .	2 .	9¾	for *A*
And	472,5	×0,0625=29,53125	=29 .	10 .	7½	for *B*
	825,	×0,0625=51,5625	=51 .	11 .	3	for *C*
	966,	×0,0625=60,375	=60 .	7 .	6	for *D*
	708,7	×0,0625=44,29375	=44 .	5 .	10½	for *E*
	880,25	×0,0625=55,015625	=55 .	0 .	3¾	for *F*
		The whole Gain	=258 .	18 .	4½	

These few *Examples* being well understood, are sufficient to shew the whole Business of Fellowship, *&c.*

Sect. 3. *Of* Bartering.

When *Merchants*, or *Tradesmen*, Exchange one Commodity for another, it's called *Bartering*; and the only difficulty in this way of Dealing, lies in the due proportioning the Commodities to be Exchanged; So, as that neither Party sustain Loss.

Question 1. Two *Merchants*, *A.* and *B.* *Barter*; *A.* would Exchange 5 *C.* 3 *qrs.* 14 *pound* of *Pepper*, which is worth 3*l.* 10*s.* *per C.* with *B.* for *Cotton*, worth 10 *d.* *per pound* weight; How much *Cotton* must *B.* give to *A.* for his *Pepper*?

Note, *In order to the Resolving of this* Question, *(and all other* Questions *of this Nature) you must First find, by the* Rule of Three, *(or otherwise) the true Value of that Commodity whose Quantity is given*; *(which in this* Question *is Pepper.) And then find how much of the other Commodity will Amount to that Sum, at the Rate proposed.*

First 5 *C.* 3 *qrs.* 14 lb = 5,875 } in *Decimals.*
And 3 *l.* 10 *s.* 0 *d.* = 3,5 }

Then 1 : 3,5 :: 5,875 : 20,5625 = 20 *l.* 11 *s.* 3 *d.* the true value of the *Pepper*.

Next, it's easy to conceive, that *A.* ought to have as much *Cotton* at 10 *d.* *per pound*, as will Amount to 20 *l.* 11 *s.* 3 *d.* which may be thus found;

10 *d.* : 1 lb :: 20 *l.* 11 *s.* 3 *d.* = 4235 *d.* ; 493,5 lb.

That

That is, 4 *C.* 1 *qr.* 17 $\frac{1}{2}$ *pound* of *Cotton.* And ſo much *B* muſt give to *A* in exchange for his 5 *C.* 3 *qrs.* 14 *pound* of *Pepper.*

Queſtion 2. Two *Merchants A* and *B Barter* thus; *A* hath 86 *Yards* of *Broad Cloth* worth 9 *s.* 2 *d. per Yard.* ready Money; but in *Barter* he will have 11 *s. per Yard. B* hath *Shalloon* worth 2 *s.* 1 *d. per Yard* ready *Money*; 'Tis required to find how many *Yards* of the *Shalloon B* muſt give to *A* for his *Cloth*, to make his gain in the *Barter* Equal to that of *A*'s.

The Method of reſolving this, and the like Queſtions, differs a little from the laſt Caſe; for in this you muſt Firſt find what Advance *B* ought to make *per Yard* upon his *Skalloon*, in proportion to what *A* hath done upon a *Yard* of his *Cloth.*

Thus { *s. d.* *d.* *s.* *d.* *s. d.* *d.* *s. d. d.*
9. 2 = 110 : 11 = 132 :: 2. 1 = 25 : 2. 6 = 30

the Advanced Price for a *Yard* of *B*'s *Shalloon.* Then proceed as before in the laſt *Example.*

Thus 1 *Yard* : 11 *s.* :: 86 *Yards* : 946 *s.* = 47 *l.* 6 *s.* the Advanced Value of all the *Cloth.*

Next, If 2 *s.* 6 *d.* will buy one *Yard* of *Shalloon*, at its Advanced Price, How many *Yards* will 47 *l.* 6 *s.* buy.

Thus 2,5 : 1 :: 946 : 378,4 *Yards.*

That is, *B* muſt give 378 $\frac{2}{5}$ *Yards* of his *Shalloon* to *A* for his 86 *Yards* of *Broad Cloth.*

Theſe Two *Examples* are ſufficient to ſhew the *Learner*, that the Method of *Bartering*, or Exchanging *Commodities* for *Commodities*, wholly depends upon a clear Underſtanding of the *Golden Rule*; which indeed is ſo called, becauſe of its *Univerſal* Uſe.

Sect. 4. *Of* Exchanging Coins.

Exchanging the *Coins* of one Country for thoſe of another is like the Buſineſs of *Bartering Commodities.* That is, it conſiſts in finding what Sum of another *Country Coin* will be Equal in value to any propoſed Sum of another *Country Coin.* And in order to perform that, it will be very Neceſſary to have a true Account at all times of the juſt Values of thoſe *Foreign Coins* which are to be Exchanged, as they are compared in value with our *Engliſh Coin.*

I ſay, at all times, becauſe the *Par* of *Exchange* (as the *Merchants* call it) differs almoſt every Day from *London* to other *Countries.* That is, it Riſes and Falls, according as *Money* is Plenty or Scarce; or according to the Time allowed for Payment of the *Money* in *Exchange*, &c.

Thoſe that deſire to be fully ſatisfied in the Common Values of *Foreign Coins*, *Weights*, *Meaſures*, &c. may find them in a Book called the *Merchants Map of Commerce*, which for Brevity ſake I have omitted Tranſcribing, and only collected theſe Few of *Coin*.

Foreign Coins.		*Engliſh Coin.* l.	s.	d.
French Coins.	A *Denier* =	0	0	$0\frac{3}{4}\frac{9}{10}$
	12 *Deniers* = 1 *Soulz* =	0	0	$0\frac{9}{10}$
	20 *Soulz* = 1 *Livre* =	0	1	6
	3 *Livres* = 1 *Crown* =	0	4	6
Low-Country Coin.	A *Stiver* =	0	0	$1\frac{1}{5}\frac{1}{6}$
	6 *Stivers* = 1 *Flemiſh Shilling* =	0	0	$7\frac{1}{5}\frac{1}{6}$
	20 *Stivers* = 1 *Gilder* =	0	2	0
	30 *Gilders* = $33\frac{1}{3}$ *Shillings* or a *Flemiſh Pound* =	1	0	0
	A *Emblem Doller* =	0	2	$3\frac{3}{5}\frac{1}{6}$
	A *Campen Doller* =	0	2	$7\frac{1}{5}\frac{1}{6}$
	A *Zealand Doller* =	0	3	0
	A *Lyons Doller* =	0	4	0
	A *Specie Doller* =	0	5	0
	Duccatoon =	0	6	$3\frac{3}{5}\frac{3}{4}$
Germany.	A *Rix Doller* of the *Empire* =	0	4	5
	A *Gilder* of *Neremberg* =	0	7	1
In Italy *and* Spain.	The *Livre* at *Leghorn* =	0	0	9
	Florence Crown Courrant =	0	5	3
	Venice Ducat de Banco =	0	4	4
	The *Currant Ducat* =	0	3	4
	The *Naple Ducat* =	0	5	0
	The *Cadiz Ducat* =	0	5	$6\frac{1}{4}$
	The *Barcelona Ducat* =	0	6	0
	The *Valentia Ducat* =	0	5	3
	The *Bergonia Ducat* =	0	4	4
	The *Portugal Teſtoon* =	0	1	3
	The Piece of *Eight* =	0	4	6

Note, The *Engliſh* generally Reckon their Exchange with other *Countries* by *Pence*, *viz.* other *Countries* value their *Crowns*, *Dollers*, or *Ducats*, &c. by *Engliſh Pence*. Except with ſome Parts of the *Low-Countries*, with whom the Exchange is in *Pounds Sterling*.

Queſt. 1. How many *Dollers* at 4 *s.* 6 *d. per Doller*, may one have for 162 *l.* 18 *s.* *Anſwer* 724 *Dollers*.

Thus

Thus 162 *l.* 18 *s.*=3258 *s.* and 4 *s.* 6 *d.*=4,5 *s.*
Then 4,5 : 1 :: 3258 : 724 the *Answer.*

Queſt. 2. How many *Saragoſa Ducats* of 5 *s.* 6 *d.* the *Ducat* may be had for 275 *Bergonia Ducats*, at 4 *s.* 4 *d.* the piece.

Anſwer 216 and 3 *s.* 8 *d.* over.

Thus 5 *s.* 6 *d.*=66 *d.* and 4 *s.* 4 *d.*=52 *d.*
Then 275×52=14300 *d.*=275 *Ducats.*
Conſequently 66) 14300 (216$\frac{2}{3}$ the *Anſwer* required.

Queſt. 3. A *Traveller* would change 233 *l.* 16 *s.* 8 *d. Sterling Money*; for *Venice Ducats* at 4 *s.* 9$\frac{1}{2}$ *d. per Ducat*; How many *Ducats* muſt he have?

Anſwer 976 *Ducats.*

Thus 4 *s.* 9$\frac{1}{2}$ *d.*=57,5 *d.* and 233 *l.* 16 *s.* 8 *d.*=56120 *d.*
Then 57,5 *d.*) 56120 *d.* (976 the *Anſwer* required.

Queſt. 4. A *Caſhier* hath received 759 *Ducats*, at 7 *s.* 6 *d. per Ducat*; And 579 *Dollers* at 4 *s.* 8 *d. per Doller*: Which he would Exchange for *Flemiſh Marks* at 14 *s.* 3 *d. per* Piece: How many ought he to have?

Anſwer 589 *Marks*, and 15 *d.* over.

For 7 *s.* 6 *d.*=90 *d.* and 4 *s.* 8 *d.*=56 *d.*
Then { 759 × 90=68310 *d.* the Value of the *Ducats.*
579 × 56=32424 *d.* the Value of the *Dollers.*
their *Sum* =100734 *d.*
And 14 *s.* 3 *d.*=171 *d.* the *Flemiſh Mark* in *Pence.*
Conſequently 171) 100734 (589 *&c.* the *Anſwer* required.

Queſt. 5. A *Bill of Exchange* was accepted at *London* for the Payment of 400 *l. Sterling*, for the like Value delivered in *Amſterdam*, at 1 *l.* 13 *s.* 6 *d.* for 1 *l. Sterling*; How much *Money* was delivered at *Amſterdam*?

Anſwer 670 *l. Flemiſh.*

For 1 *l.*=240 *d.* and 1 *l.* 13 *s.* 6 *d.*=402 *d.*
Then 240 : 402 :: 400 : 670 the *Anſwer* required.

Queſt. 6. When the *Exchange* from *Antwerp* to *London* is at 1 *l.* 14 *s.* 7 *d. Flemiſh*, for 1 *l. Sterling*; How many *Pounds Sterling* muſt be paid at *London*; to ballance 236 *l. Flemiſh* at *Antwerp*?

Anſwer 192 *l. Sterling.*

Thus 1 *l.* 4 *s.* 7 *d.*=295 *d.* and 1 *l.*=240 *d.*
Then 295 : 240 :: 236 : 192 the *Anſwer.*

Queſt. 7. A *Merchant* delivered at *London* 120 *l. Sterling* to Receive 147 *l. Flemiſh* in *Amſterdam*; How much was 1 *l. Sterling* valued at, in *Flemiſh* Money.

Anſwer 1 *l.* 4 *s.* 6 *d.*

Thus 120 : 147 : : 240 *d.* : 294 *d.*=1 *l.* 4 *s.* 6 *d.* &c.

Queſt. 8. A *Factor* hath ſold Goods at *Cadiz* for 1468 pieces of Eight; valued at 4 *s.* 6½ *d. Sterling per* Piece; How much *Sterling* Money do thoſe Pieces of Eight amount to.

Anſwer 333 *l.* 7 *s.* 2 *d.*

Thus, if 1=54,5 *d.* then 1468×54,5=80006 *d.* &c.

Queſt. 9. A *Traveller* would have an equal *Number* of *Crowns* at 5 *s.* 6 *d. per Crown*; And *Dollers* at 4 *s.* 5 *d. per* Piece; How many of each ſort may he have for 309 *l.* 8 *s.*?

Anſwer 624 of each.

Thus 309 *l.* 8 *s.*=74256 *d.*
And 5 *s.* 6 *d.*+4 *s.* 5 *d.*=119 *d.*
Then 119) 74256 (624 the *Anſwer* required.

Queſt. 10. Suppoſe I would Exchange 527 *l.* 17 *s.* 6 *d.* for *Dollers* at 4 *s.* 6 *d.* a piece, *Ducats* at 5 *s.* 8 *d.* a piece, and *Crowns* at 6 *s.* 1 *d.* a piece; and would have 2 *Dollers* for 1 *Ducat*, and 3 *Dollers* for 2 *Crowns*? How many of each ſort muſt I have?

Anſw. 927 *Dollers*, 463½ *Ducats*, and 618 *Crowns*.

For { 54 *d.*=1 *Doller.* / 68 *d.*=1 *Ducat.* / 73 *d.*=1 *Crown.* } *per Queſtion.*

And 126690 *d.*=527 *l.* 17 *s.* 6 *d.*

Now if the *Crowns*, *Dollars*, and *Ducats* were to be equal in *Number*; Then 73+54+68 muſt have been the *Diviſor*, by which 126690 muſt have been *Divided*, and the *Quotient* would have been the *Anſwer* to the *Queſtion*. As in the laſt *Example*.

But here inſtead of their *Sum*, ſuch Parts of them muſt be taken as are aſſigned or limited by the *Queſtion*; that ſo the *Number* of ſome one of them may be found.

And becauſe there muſt be { 2 *Dollers* for 1 *Ducat*, and / 3 *Dollers* for 2 *Crowns*, }

Therefore it will be but ½ of a *Ducat* for one *Doller*, and ⅔ of a *Crown* for one *Doller*.

Con-

Conſequently, $54+\frac{68}{2}:+\frac{2}{3}$ of $73=136\frac{1}{3}$ Or $\frac{410}{3}$ will be the *Diviſor* to find the *Number* of *Dollers*.

Thus $\frac{410}{3}$) 126690 (927 the *Number* of *Dollers*.

Then $\frac{1}{2}$ of $927=463\frac{1}{2}$ is the *Number* of *Ducats*.

And $\frac{2}{3}$ of $927=618$ is the *Number* of *Crowns*.

Or if you pleaſe you may form *Diviſors* to find either the *Ducats* or *Crowns* firſt: For if it be 2 *Dollers* for 1 *Ducat*, and 3 *Dollars* for 2 *Crowns*, as before;

Then will 6 *Dollers* be for 3 *Ducats*, and 6 *Dollers* for 4 *Crowns*.

Therefore, $\left\{\begin{array}{l}\frac{3}{2}\text{ of a }Doller\\ \frac{3}{4}\text{ of a }Ducat\end{array}\right\}$ will be for 1 *Crown*.

Conſequently, $\frac{3}{2}$ of $54:+\frac{3}{4}$ of $68:+73=205$ will be the *Diviſor* to find the *Crowns* firſt, *&c.*

Queſt. 11. A *Caſhire* is to Receive 500 *l.* He is offered *Crowns* at 6 *s.* $1\frac{1}{2}$ *d. per Crown*, which are worth but 6 *s.* Or he may have *Dollers* at 4 *s.* 5 *d.* the piece, which are worth but 4 *s.* 4 *d.* Which of theſe ſhall he receive to have the leaſt Loſs? And how much will he loſe in Payment?

1 $\left\{\begin{array}{l}1\ Crown=72\ d.\\ 1\ Doller=52\ d.\end{array}\right\}$ according to the true Values.

2 $\left\{\begin{array}{l}1\ Crown=73,5\ d.\\ 1\ Doller=53\ \ d.\end{array}\right\}$ the advanced Values.

Now to find which will be the leaſt Loſs; find what the advanced Value of a *Doller* ought to be in proportion to that of a *Crown*.

Thus 72 : 73,5 :: 52 : 53,22 *&c.* But he may have *Dollers* at 53 *d. per* piece, therefore the Payment in *Dollers* will be the leaſt Loſs; *viz.* 53 is leſs than 53,22 *&c.*

Next to find what the whole Loſs will be, *Divide* 120000 *d.* =500 *l.* by 52. and 53. The Difference of their *Quotients* will be the Loſs.

Thus 52) 120000 ($2307\frac{9}{13}$. And 53) 120000 ($2264\frac{8}{53}$.

Then $2307\frac{9}{13}-2264\frac{8}{53}=43\frac{573}{689}$ *Dollers* at 4 *s.* 4 *d.* is the Loſs; *viz.* 9 *l.* 8 *s.* $10\frac{226}{689}$ *d.*

There are other ways of anſwering the laſt *Queſtion*, but this I take to be the eaſieſt.

Queſt. 12. Suppoſe I exchange 4 *l.* 10 *s.* 10 *d.* for 11 *Crowns* and 7 *Dollers*; And at another time I have 4 *Crowns* and 3 *Dollers*;

Dollers for 1 *l.* 15 *s.* each being of the ſame Value with th firſt. What is the Value of a *Crown*, and of a *Doller*?

First 11 *Crowns*+7 *Dollers*=1090 *d.*
Second 2 *Crowns*+3 *Dollers*= 420 *d.* } by the *Queſtion.*

Then in order to find the Value of 1 Crown, you muſt caſt o the *Dollers* by making them of the ſame *Number*; Thus,

33 *Crowns*+21 *Dollers*=3270 *d.* the firſt *Multipl.* with 3.
28 *Crowns*+21 *Dollers*=2940 *d.* the ſecond *Multipl.* with 7.

Then 5 *Crowns*= 330 *d.* being their Difference.
Conſequently 5) 330 (66= 5 *s.* 6 *d.* is the Value of 1 *Crown.*
And 4 *Crowns*=264 *d.*
Then will 3 *Dollers*=420 *d.*—264 *d.*=156 *d.*
Conſequently 3) 156 (52 *d.* = 4 *s.* 4 *d.* the Value of 1 *Doller.*

CHAP. IX.

Of Alligation.

WHEN it is required to mix ſeveral Sorts of Ingredien together; As different ſorts of *Corn*, *Wines*, *Wooll*, *Spice* or *Metals*; or to compoſe *Medicines*, &c. the Method of pr portioning ſuch Mixtures, is called the *Rule of Alligation*; an is Divided into two Parts or Branches; called *Medial* an *Alternate.*

Sect. 1. *Of* Alligation Medial.

Alligation Medial, is that by which the Mean Rate or Pri of any Mixture is found, when the particular Quantities of th Mixtures and Rates are given: And is thus performed.

First find the *Sum* of all the Quantities propoſed to be mix'd And alſo the *Sum* of all their particular Rates.

Then the Proportion will be,

Rule. { *As the Sum of all the Quantities: Is to the Sum all their Rates:: So is any Part of the Mixtur To the Mean Rate or Price of that Part.*

Queſt. 1. Suppoſe 15 *Buſhels* of *Wheat* at 5 *s.* the *Buſh* and 12 *Buſhels* of *Rye* at 3*s.* 6*d.* the *Buſhel*, were mix'd together Wh

hat is the Mean Rate or Price, it may be ſold for a *Buſhel*, thout Loſs or Gain ?

This *Queſtion* prepared as Directed above will ſtand

hus { 15 *Buſhels* of *Wheat* at 5 *s. per Buſhel*, comes to 900 *d.*
12 *Buſhels* of *Rye* at 3 *s.* 6 *d.* each, comes to 504 *d.*

27=their *Sum*. And their Total Value=1404 *d.*

Then 27 *Buſhels* : 1404 *d.* : : 1 *Buſhel* : 52 *d.*=4 *s.* 4 *d.* the *nſwer* required.

Queſt. 2. A *Grocer* Mixeth 36 *Pounds* of *Tobacco*, worth 1 *s.* *d.* a *Pound*, with 12 *Pounds* of another ſort at 2 *s.* a *Pound.* d 12 *Pounds* of a third ſort at 1 *s.* 10 *d.* the *Pound.* How may ſell the *Mixture per Pound* ?

	lb		*s.*	*d.*		*d.*
Firſt	36	at	1	6	*per Pound* Amounts to	648
	12	at	2	0		288
	12	at	1	10		264

60=the *Number* of *Pounds* ; their Value=1200

Then 60 lb : 1200 *d.* : : 1 lb : 20 *d.* = 1 *s.* 8 *d.* the *Anſwer* quired.

Queſt. 3. A *Vintner* Mixeth 31 *Gallons* and half of *Malaga* ck worth 7 *s.* 6 *d.* the *Gallon* ; with 18 *Gallons* of *Canary* at 9 *d.* the *Gallon* ; 13 *Gallons* and half of *Sherry* at 5 *s.* the *llon* ; And 27 *Gallons* of *White Wine* at 4 *s.* 3 *d.* the *Gallon.* is required to find what one *Gallon* of this Mixture is worth.

	Gal.		*s.*	*d.*		*Pence.*
Firſt,	31½	at	7	6	*per Gallon* comes to	2835
	18	at	6	9		1458
	13⅓	at	5	0		810
	27	at	4	3		1377

90=the *Number* of *Gal.* Their Value=6480

Then 90 : 6480 : : 1 : 72 *d.*=6 *s.* the *Rate* or *Price* of one *llon*, as was required.

The *Proof* of all *Operations* in theſe ſort of *Mixtures*, is done comparing the Value of all the Mixture, being ſold at the an Rate ; with the Total Value of all the particular Quanti-s, ſuppoſing they had been ſold at their reſpective Rates un-x'd ; if thoſe *Sums* are equal the Work is true.

Sect.

Sect. 2. *Of* Alligation Alternate.

Alligation Alternate, is that by which the particular Quantit of every Ingredient concern'd in any *Mixture* are found ; wh the *particular Rates* of every one of thoſe Ingredients, and *mean Rate* are given ; being (as it were) the Converſe to *Alli tion Medial* ; as will appear by the following *Operations*, wh admits of three *Caſes*.

Caſe I. The *Particular Rates* of any Ingredients propoſed be mixed, and the *Mean Rate* of the whole *Mixture* be given. To find how much of each Ingredient is requiſite compoſe the *Mixture* ; when the whole Quantity, or any p thereof is not *Limited*.

Queſt. 1. How much *Wheat* at 55. the *Buſhel*, and *Rye* 3 s. 6 *d.* the *Buſhel*, will compoſe a *Mixture* that may be ſ for 45. 4 *d.* the *Buſhel* ?

Note, *In all* Queſtions *of this* Nature, *it will be convenient place the* Mean Rate *ſo, as that it may be eaſily compared* w *the* Particular Rates, *in order to find every one of their Di rences from the* Mean Rate, *by inſpection only*.

Thus, the *Mean Rate* = 52 *d.* { *Wheat* 60 *d.* / *Rye* 42 *d.*

Then take the ſeveral Differences between the Mean Ra *and the* Particular Rates ; *Setting down thoſe Differences Al nately, and they will be the Quantities required.*

Thus 52 { 60 / 42 } { 10 = 52 − 42 / 8 = 60 − 52 }

That is 52 − 42 = 10 for the quantity of *Wheat*.

And 60 − 52 = 8 for the quantity of *Rye*, that will co poſe the *Mixture* required.

The *Proof* by *Alligation Medial*.

Add { 10 *Buſhels* of *Wheat* at 60 *d. per Buſhel* = 600 *d.* / 8 *Buſhels* of *Rye* at 42 *d. per Buſhel* = 336 *d.* }

18 = the *Number* of *Buſhels*. = 936 *d.*

Then 18 : 936 :: 1 : 52 *d.* = 4 s. 4 *d.* the *Mean Rate*.

Note, Atho' 10 and 8 do *Anſwer* the *Queſtion*, as plai appears by the *Proof* ; yet they are not the only Two *Numbe* for this *Queſtion*, and all others of this kind, will admit of rious *Anſwers*, and all in whole *Numbers* ; for any Two *Numb* that are in the ſame Proportion to one another, as 10 is to will as truly *Anſwer* the *Queſtion*.

Viz. 10 : 8 :: { 5 : 4 / 15 : 12 / 20 : 16 / 25 : 20 } *&c. ad infinitum.*

Queſt. 2. A *Grocer* would mix Three ſorts of *Tobacco* together, z. One Sort of 18 *d. per* ℔. another *Sort* of 22 *d. per* ℔. and Third *Sort* of 2 *s.* the ℔. How much of each *Sort* muſt he ke, that the whole *Mixture* may be ſold for 20 *d.* the *Pound*? Having ſet down the given *Rates*, as before: Then find each their Differences from the propoſed *Mean Rate*; And place oſe Differences *Alternately*. Thus,

Mean Rate 20 { 18 / 22 / 24 } { 4+2=24—20 and 22—20 / 2=20—18 / 2=20—18 }

Theſe Differences, *viz.* 6 . 2 . 2 are the Quantities *Required.*

Proof { 6 ℔ of *Tobacco* at 18 *d. per Pound* comes to 108 / 2 ℔ at 22 *d.* the *Pound* comes to 44 / 2 ℔ at 24 *d.* the *Pound* comes to 48 } *d.*

10=the *Number* of *Pounds.* Their value=200 *d.*

Then 10) 200 (20 the *Mean Rate.*

Or indeed any three *Numbers* that have the ſame *Ratio* to one nother as 6 . & 2 have, will *Anſwer* the *Queſtion.*

That is 6 : 2 :: { 9 : 3 / 12 : 4 / 15 : 5 } *&c.*

But if only one of the three given *Rates* had been greater han the *Mean Rate*; As ſuppoſe 14 *d. per Pound*, 18 *d. per Pound*, and 24 *d. per Pound.* And the *Mean Rate* 20 *d.* as efore. Then their Differences muſt have been placed,

Thus, 20 { 14 / 18 / 24 } { 4 / 4 / 6+2 } *&c.* As before.

Queſt. 3. A *Vintner* would make a *Mixture* of *Malaga*, orth 7 *s.* 6 *d. per Gallon*, with *Canary* at 6 *s.* 9 *d. per Gallon*, *herry* at 5 *s. per Gallon*, and *White Wine* at 4 *s.* 3 *d. per allon*; What Quantity of each ſort muſt he take, that the *ixture* may be ſold for 6 *s. per Gallon*?

In all *Queſtions* of this kind, wherein it is required to mix our Things together, Two of them having their Prices *Greater*,

and *Two Leſſer* than the *Mean Rate*; you muſt always *Alligate* or *Compare* a *Greater* and *Leſſer Price* with the *Mean Price* ſetting down their Differences *Alternately*, as in the *Firſt Example* of this *Section*.

Thus, *Mean Rate*=72 *d.*

Malaga	90 *d.*	21	=72—51
White	51 *d.*	18	=90—72
Sherry	60 *d.*	9	=81—72
Canary	81 *d.*	12	=72—60

Hence 21 *Gallons* of *Malaga*, 12 of *Canary*, 9 of *Sherry*, and 12 of *White will* compoſe the *Mixture required*.

Or thus, 72

Malaga	90 *d.*	12	*Malaga*
Sherry	60 *d.*	18	*Sherry*
Canary	81 *d.*	21	*Canary*
White	51 *d.*	9	*White*

will, *&c.*

Either of theſe *Mixtures* equally *Anſwer* the *Queſtion*, which may be eaſily try'd as before in the *Laſt*, *&c.*

Caſe II. The *Particular Rates* of all the Ingredients propoſed to be mix'd, the *Mean Rate* of the whole *Mixture*, and any one of the Quantities to be mix'd being given. Thence to find how much of every one of the other Ingredients is requiſite to compoſe the Mixture.

Note, This is uſually called *Alligation Partial.*

Queſt. 4. How much *Wheat* at 5 *s.* the *Buſhel*, muſt be mix'd with 12 *Buſhels* of *Rye* at 3 *s.* 6 *d.* a *Buſhel*; That the whole Mixture may be ſold for 4 *s.* 4 *d.* the *Buſhel*?

In this Caſe you muſt ſet down all the *Particular Rates*, with the *Mean Rate*, and find their Differences juſt as before; without any regard had to the Quantity given.

Thus, *Mean Rate* 52 *d.*

Wheat	60 *d.*	10
Rye	42 *d.*	8

Then *As the Quantity found by the Differences of the ſame* Name *with the Quantity given: Is to the Quantity given :: So is any of the other Quantities found by the Differences: To the Quantity of its* Name.

Thus 8 : 12 :: 10 : 15. the *Quantity* or *Number* of *Buſhels* of *Wheat* required.

Queſt. 5. How much *Malaga* at 7 *s.* 6 *d.* the *Gallon*, *Sherry* at 5 *s.* the *Gallon*, and *White Wine* at 4 *s.* 3 *d.* the *Gallon*, muſt be mix'd with 18 *Gallons* of *Canary* at 6 *s.* 9 *d.* the *Gallon*; That the whole Mixture may be ſold for 6 *s.* the *Gallon*?

The

The *Terms* being ſet down, &c. as before, will ſtand

Thus, *Mean Rate* 72 *d.* { *Malaga* 90 *d.* } { 21
White 51 *d.* } { 18
Sherry 60 *d.* } { 9
Canary 81 *d.* } { 12

Then, As 12 : 18 :: { 21 : $31\frac{1}{2}$ *Gallons of Malaga.*
18 : 27 *Gallons of White.*
9 : $13\frac{1}{2}$ *Gallons of Sherry.*

That is, $31\frac{1}{2}$ *Gallons of Malaga*, 27 of *White Wine*, and $13\frac{1}{2}$ *Sherry*, being mix'd with 18 *Gallons of Canary*, will make e Mixture required.

Or thus, 72 { *Malaga* 90 } { 12
Sherry 60 } { 18
Canary 81 } { 21
White 51 } { 9

Then, As 21 : 18 :: { 12 : $10\frac{6}{21}$ *the Malaga*
18 : $15\frac{9}{21}$ *the Sherry*
9 : $7\frac{15}{21}$ *the White Wine.* } &c.

Proof

	Gallons,		*Pence.*
$10\frac{6}{21}$	*at* 90 *d. each* =		$925\frac{15}{21}$
$15\frac{9}{21}$	*at* 60 *d. each* =		$925\frac{1}{21}$
$7\frac{15}{21}$	*at* 51 *d. each* =		$393\frac{9}{21}$
18	*at* 81 *d. each* =		1458
Sum $51\frac{9}{21}$		*Value* =	$3702\frac{18}{21}$

Then $51\frac{9}{21}$) $3702\frac{18}{21}$ (72 *d.* = 6 *s.* the *Mean Rate.*

Therefore the *Quantities* are as truly aſſigned here, as in the ſt Work.

Caſe III. The *Particular Rates* of all the Ingredients propoſed be mix'd; and the *Sum* of all their Quantities, with the *Mean ate* of that *Sum* being given; To find the particular *Quantities* the Mixture.

This is called *Alligation Total*, and is thus performed.

et down all the *Particular Rates*, with the *Mean Rate*, and nd their Differences, as before: Add together all the Differences to one *Sum*;

Then { *As the* Sum *of all the Differences* : *Is to the* Sum *of all the Quantities given* :: *So is every particular Difference* : *To its particular Quantity.*

Queſt. 6. Let it be required to mix *Wheat* at 5 *s.* the *Buſhel*, ith *Rye* at 3 *s.* 6 *d.* the *Buſhel*; So as that the whole *Quantity* ay be 27 *Buſhels*, to be Sold for 4 *s.* 4 *d.* a *Buſhel*; What *uantity* of each muſt be taken to make up the Mixture?

Mean Rate 52 { *Wheat* 60 *d.* } { 10
Rye 42 *d.* } { 8

18 = their *Sum*

Then 18 : 27 :: { 10 : 15 / 8 : 12 } The *Quantities* required.

Queſtion 7. Suppoſe it were required to mix *Malaga* at 7 *s.* 6 *d.* the *Gallon*, with *Canary* at 6 *s.* 9 *d.* the *Gallon*; *Sherry* at 5 *s.* the *Gallon*, and *White Wine* at 4 *s.* 3 *d.* the *Gallon*; So as that the whole Mixture may be 90 *Gallons*; to be ſold for 6 *s.* the *Gallon*: How much of each ſort will compoſe that Mixture?

Mean Rate = 72 *d.* { *Malaga* 90 } { 21
White 51 } { 18
Canary 81 } { 9
Sherry 60 } { 12

60 = their *Sum.*

Then { 60 : 90 :: 21 : 31½ *the Gallons of Malaga.*
60 : 90 :: 18 : 27 *the Gallons of White Wine.*
60 : 90 :: 9 : 13½ *the Gallons of Sherry.*
60 : 90 :: 12 : 18 *the Gallons of Canary.*

Or thus, 72 { *Malaga* 90 } { 12
Sherry 60 } { 18
Canary 81 } { 21
White 51 } { 9

60 their *Sum.*

Then { 60 : 90 :: 12 : 18 *Gallons of Malaga.*
60 : 90 :: 18 : 27 *Gallons of Sherry.*
60 : 90 :: 21 : 31½ *Gallons of Canary*
60 : 90 :: 9 : 13½ *Gallons of White Wine.*

Either of theſe ways do equally *Anſwer* the *Queſtion*, as may be eaſily tried by *Alligation Medial*. As before, *&c.*

Note, *The Work of theſe Proportions may be much ſhortened (eſpecially when there are many Ingredients to be mix'd) if you obſerve the ſame Method as was propoſed in the* Rule of Fellowſhip, *page* 99, *&c.*

I have made uſe of the very ſame *Examples* both in *Alligation Medial*, and *Alternate*, throughout the three *Caſes*; being, as I preſume, much better than if they had been different ones; becauſe the Learner may (if he conſider a little on them) eaſily perceive, not only the Difference between the two

wo *Rules*, but alſo wherein the chief Difference of each Caſe n the *Alternate Rule* depends, *&c.* Not but that I could have nſerted many various *Examples*, as alſo the Manner of Compoing Medicines, *&c.* which, for Brevity ſake, I have omitted nd refer thoſe that deſire to ſee into that Buſineſs to Sir *Jonas More*'s *Arithmetick*, wherein he will find it largely handled. And ſo I ſhall conclude with *Alligation Alternate*, which altho' gives true *Anſwers* to *Queſtions* of that kind, with ſome little ariety, according as the Ingredients are More or Leſs in *Number*; s appears by the foregoing *Examples*; Yet it will not give all he *Anſwers* as ſuch *Queſtions* are capable of, nor perhaps thoſe which ſuit beſt with the preſent Occaſion: Nor can this Imperfection be remedied by common *Arithmetick*; but by an *Algebraick* way of Arguing it may; whereby all the poſſible *Anſwers* o any *Queſtion* may be clearly and eaſily diſcover'd; As ſhall e ſhew'd further on in the Second Part.

CHAP. X.

Of Metals *and their* Specifick Gravities, *&c.*

Sect. 1. *Of* Gold *and* Silver.

PURE Gold, free from Mixture of other Metals, uſually called *Fine Gold*, is of ſuch a Nature and Purity that it will endure the *Fire* without waſting, although it were kept continually Melted: And therefore ſome of the *Ancient Philoſophers* have ſuppoſed the *Sun* to be a *Globe* of Liquid or Melted *Gold*.

Silver having not the Purity of *Gold*, will not indure the *Fire* like it; Yet *Fine Silver* will waſte but a very little by being in the *Fire* any reaſonable time; whereas *Copper*, *Tin*, *Lead*, *&c.* will not only waſte, but may be *calcin'd* or *burnt* to a Powder.

Both *Gold* and *Silver* in their Purity, are ſo very flexible or ſoft (like new *Lead*, *&c.*) that they are not ſo uſeful either in *Coin*, or otherwiſe (except to beat in *Leaf-Gold* or *Silver*) as when they are allay'd, or mix'd and harden'd with *Copper* or *Braſs*. And altho' moſt Places differ more or leſs in the *Quantity* of ſuch *Allay*, yet in *England* it is generally agreed on, that,

Stand-

Standard for Gold.

22 Carracts of *Fine Gold*, and 2 Carracts of *Copper*, being melted together ſhall be eſteemed the true Standard for *Gold Coin*, &c. *The* French *and* Spaniſh *Gold being very near of the ſame Standard.*

That is, if any *Quantity* or *Weight* of *Fine Gold*, be *divided* into Twenty Four equal Parts, and 22 of thoſe Parts be mix'd with 2 of the like Parts of *Copper*; that Mixture is called *Standard Gold.*

Whence you may obſerve, that a *Carract* is not any certain *Quantity* or *Weight*, but $\frac{1}{24}$ part of any *Quantity* or *Weight*; and the *Minters* and *Goldſmiths Divide* it into 4 equal Parts, which they call *Grains* of a *Carract*; alſo they *ſubdivide* one of thoſe *Grains*, into *Halves*, *Quarters*, &c.

Standard for Silver.

Eleven Ounces and *Two Penny-weight* of Fine *Silver*, and *Eighteen Penny-weight* of *Copper* being melted together, is eſteem'd the true Standard for *Silver Coin*, called *Sterling Silver.* And ſo in *Proportion* for a greater or leſſer *Quantity*; which is a Leſs *Proportion* of *Allay* for *Silver*, than the other is for *Gold.*

Note, When either *Silver* or *Gold* is Finer than *Standard*, it's called *Better*; if Coarſer, it's called *Worſe*; and that Betterneſs or Worſeneſs, is reckoned by *Carracts* and *Grains* of a *Carract* in *Gold*; and by *Penny-weights* in *Silver*, and is thus diſcovered: The *Goldſmiths* or *Refiners*, &c, do take a ſmall *Quantity* of ſuch *Gold* as they intend to Try (which they call making an *Aſſay*) and weigh it very exactly, then they put it into a *Crucible*, and melt it in a ſtrong Fire, ſo long that if there be any *Copper*, or other *Allay mixt* with it, that *Allay* may be conſum'd or burnt away: When it's cold they weigh it very *exactly* again, and if it have loſt nothing of its firſt Weight, they conclude it is *Fine Gold*, but if the Loſs be $\frac{1}{24}$ *Part*, they call it 23 *Carracts Fine*, or one *Carract better* than Standard: If it have loſt $\frac{2}{24}$ *Parts*, it's 22 *Carracts fine*, or Standard: If $\frac{3}{24}$ *Parts*, it's ſaid to be 21 *Carracts fine*, or rather one *Carract worſe* than Standard, and ſo in Proportion as it happens to be *Better* or *Worſe.*

In the ſame Manner they make their Aſſay on *Silver*, only they compute its Loſs by *Penny-weights*, &c.

The Author of the *Preſent State of England*, mention'd before (*page* 32.) ſays,

'That

'That the *English Coin* may not want neither the Purity nor Weight required, it is most wisely and carefully provided, that once every Year the chief Officers of the *Mint* appear before the Lords of the Council in the *Star-Chamber* at *Westminster*, with some Pieces of all sorts of Moneys Coined the foregoing Year, taken at adventure out of the *Mint*, and kept under several Locks, by several Persons, 'till that Appearance, and then by a Jury of 24 able *Goldsmiths*, in the Presence of the said Lords, every Piece is most exactly Weighed and Assay'd.

This if it were constantly practised would keep our Coin to its true Standard, *&c.*

Many pretty *Questions* may be started concerning the Fineness of *Gold* and *Silver*, &c.

Example 1.

If an *Ingot* of *Silver* weighing 787 *oz.* 14 *pwt.* 6 *grains*, be 11 *oz.* 6 *pwt.* fine; How much fine *Silver* is there in it, and what amounts it to, at 5 *s.* $1\frac{1}{2}$ *d.* the *Ounce*?

This *Ingot* is *better* than Standard by 4 *pwt.*. For 11 *oz.* *pwt.* = 222 *pwt.* the fine *Silver* in 12 *oz.* of Standard. But 1 *oz.* 6 *pwt.* = 226 *pwt.* the fine *Silver* in 12 *oz.* according to the *Question.*

First 787 *oz.* 14 *pwt.* 6 *grains*=378102 *Grains.*

And 12 *oz.*=240 *pwt.*

Then, As 240 : 226 :: 378102 : 356046=741 *oz.* 15 *pwt.* $\frac{1}{20}$ *gr.* the fine *Silver* in that *Ingot.*

Which at 5 *s.* $1\frac{1}{2}$ *d.* the *Ounce*, amounts to 190 *l.* 1 *s.* 6 *d.* and near a *half-penny.*

Example 2.

If an *Ingot* of *Gold* weighing 115 *oz.* 13 *pwt.* 18 *grains*; be $\frac{1}{4}$ of a *grain* worse than Standard: How much Standard *Gold* is there in it, and what comes it to at 3 *l.* 11 *s.* an *Ounce*?

First 115 *oz.* 13 *pwt.* 18 *gr.*=55530 *Grains Troy.*
Then 24) 55530 (2313,75=a *Carrat* of that *Quantity.*
And 4) 2313,75 (578,4375=a *Grain* of that *Carrat.*
Consequently 4) 578,4375 (144,609375=$\frac{1}{4}$ of a *grain.*

Again, 2313,75×22=50902,5 ought to be the fine *Gold* in that *Ingot*, if it had been Standard:

But

But 50902,5—144,609375=50757,890625 is the *Quantity* of fine *Gold* according to the *Queſtion*.

Therefore 509025 : 50757,890625 :: 55530 : 55372,244 &c. *grains* = 115 *oz.* 7 *pwt.* 4,244 &c. *Grains Troy*, being the *Quantity* of Standard *Gold* in that *Ingot*. As was required.

Next for the Value of it, at 3 *l.* 11 *s. per Ounce*; 1 *oz.*=480 *grains*. And 3 *l.* 11 *s.*=71 *s.*

Conſequently 480 : 71 :: 55372,244 &c. : 8190,4777 &c. =409 *l.* 10 *s.* $5\frac{3}{4}$ *d.* very near; being the Value of that *Ingot*. As was required.

Or the laſt *Queſtion* may be otherwiſe wrought thus; 115 *oz.* 13 *pwt.* 18 *grains*=115,6875. And $\frac{1}{4}$ of a *Grain*, of a *Carrat* is $\frac{1}{16}$ (*viz.* the $\frac{1}{4}$ of $\frac{1}{4}$)

Then $22-\frac{1}{16}=21\frac{15}{16}=21,9375$.

Conſequently 22 : 21,9375 :: 115,6875 : 115,358842 &c. =115 *oz.* 7 *pwt.* 4,244 *grains*, &c. As before.

Next for the Value, As 1 : 3,55 :: 115,358842 : 409,52388 =409 *l.* 10 *s.* $5\frac{3}{4}$ *d.* very near. As before.

Sect. 2. *The* Specifick Gravity *of* Metals, &c.

I take an Enquiry made about the different *Gravities*, or *Weights* of *Metals*, and other Bodies, to be (not only a Work of Curioſity, but alſo) of very good Uſe upon many Occaſions. Therefore ſeveral Authors have given us ſuch *Proportions*, or *Difference* of their *Weights*, as they are ſaid to have one to another; ſuppoſing every one of them to be of the ſame *Magnitude* or *Bigneſs*. Some of which I ſhall here inſert.

1. *Henry Van Etten*, in his *Mathematical Recreations*, Printed *Anno* 1633, ſets down the *Proportion* of their *Weights*. Thus,

Gold 1875 . *Lead* 1165 . *Silver* 1040 . *Copper* 910 . *Iron* 810 . *Tin* 750 . *Water* 100.

2. One *Alſted*, in his *Encyclopædia*, Printed 1649, hath them Thus.

Gold 1875 . *Quickſilver* 1500 . *Lead* 1165 . *Silver* 1040 . *Copper* 910 . *Iron* 806 . *Tin* 750 . *Honey* 150 . *Water* 100 . *Oil* 90. Theſe ſeem to be taken from thoſe of *Van Etten's*, with ſome *Additions* only.

3. The Ingenious Mr. *Oughtred*, in his *Circles of Proportion*, Printed *Anno* 1660, hath their *Proportions* according to the *Experiments* of one *Marinus Ghetaldi*, in his Tract called *Archimedes Promotus*. Thus,

Gold 3990 . *Quickſilver* 2850 . *Lead* 2415 . *Silver* 2170 . *Braſs* 1890 . *Iron* 1680 . *Tin* 1554.

4. In

4. In the Philosophical Transactions, *(Number* 169 and 199*)* here is an Account of a great many Experiments of this kind; from whence I collected these following, *viz. Gold* 18888. *Mercury* 14019. *Lead* 11343. *Silver* 11087. *Copper* 8843. *Hammer'd Brass* 8349. *Cast Brass* 8100. *Steel* 7852. *Iron* 7643. *Tin* 7321. *Pump-water* 1000.

These last Proportions being approved of and published by Order of the *Royal Society* seem to be unquestionably true: Nevertheless, because they differ so much from the before-mentioned *(and those from one another)* I have for my own satisfaction made several Experiments of that kind: And have *(I presume)* obtained the Proportions of *Weight* that one body bears to another, of the same Bulk or Magnitude, as nicely as the Nature of such Matter, as may be contracted or brought into a lesser Body *(viz.* either by Drying, or Hammering, or otherwise) will admit of; which are as followeth.

		Ounces Troy		*Ounces Averd.*
A *Cubick Inch* of	*Fine Gold*, is	10,359273	=	11,365602
	Standard Gold,	9,962625	=	10,930422
	Quicksilver,	7,384411	=	8,101753
	Lead,	5,984010	=	6,553885
	Fine Silver,	5,850035	=	6,418324
	Standard Silver,	5,556769	=	6,096569
	Rose Copper,	4,747121	=	5,208369
	Plate Brass,	4,404273	=	4,832116
	Cast Brass,	4,272409	=	4,630300
	Steel,	4,142127	=	4,544505
	Common Iron,	4,031361	=	4,422979
	Block Tin,	3,861519	=	4,236638
	Fine Marble,	1,429411	=	1,568859
	Common Glass,	1,360841	=	1,493037
	Alabaster,	0,988456	=	1,084477
	Dry Ivory	0,962083	=	1,055542
	Dry Box-wood	0,543282	=	0,596057
	Sea Water,	0,542742	=	0,594894
	Common Clear Water,	0,527458	=	0,578697
	Red Wine,	0,523766	=	0,574646
	Proof Spirits or Brandy	0,489268	=	0,536796
	Sound Dry Oak,	0,489008	=	0,536569
	Linseed Oil,	0,491591	=	0,539345
	Oil Olive,	0,481569	=	0,528350

In this *Table* you have the *Specifick Gravity* or *Weight* of a *Cubick Inch*, of various ſorts of Bodies, both in *Troy Ounces* and *Averdupois Ounces*, and *Decimal Parts* of an *Ounce*, which I can aſſure you required more Charge, Care, and Trouble, to find out nicely, than I was at firſt aware of.

Now from hence it will be eaſy to determine the *Weight* of any propoſed Quantity, of the ſame Matter and Kind with thoſe in the *Table*; its Solid Content being given in *Cubick Inches*. For it is plain, that if the *Number* of *Cubick Inches* contained in any given Quantity, be *Multiplied* with the *Tabular Weight* of one *Inch (of the ſame kind of Matter)* the *Product* will be the *Weight* of that Quantity in *Ounces*, &c.

Example.

Suppoſe it were required to find the *Weight* of a piece of *Marble*, containing three *Solid Feet*, and 40 *Cubick Inches*.

Firſt 1728×3=5184 the *Cubick Inches* in 3 *Solid Feet*.
And 5184+40=5224 the *Number* of *Cubick Inches* in the piece of *Marble*.
Then 5224×1,429411=7410,066624 *Ounces Troy*.
Or 5224×1,568859=8195,719416 *Ounces Averdupois*.

The *Weight* of that piece of *Marble*, in *Ounces*, &c. which is eaſily brought into *Pounds*, &c. The like for any of the reſt.

The Converſe of this Work is as eaſy; *viz.* if the *Weight* of any propoſed Quantity be given, thence to find the *Solid* Content of that Quantity in *Cubick Inches*, &c.

Thus, *Divide* the given *Weight* of the propoſed Quantity *(it being firſt Reduced into Ounces*, &c.*)* by the *Tabular Weight* of one *Inch (of the ſame kind of Matter)* and the *Quotient* will be the *Number* of *Cubick Inches* contained in that Quantity.

Note, If you would find what *Weight* any Quantity of thoſe *Bodies* mentioned in the *Table* will have, when it is immerſed or put into Water, you muſt *Subſtract* the *Weight* of an equal Quantity of Water (with that of the Body) from the *Weight* of the propoſed Body (if it be heavier than Water) and there will *Remain* the *Weight* required. As for Inſtance,

A *Cubick Inch* of *Lead* =5,984010
A *Cubick Inch* of *Water*=0,542742 } *Ounces Troy*, &c.

their Difference is,=5,441268 the *Weight* of a *Cubick Inch* of *Lead* in the Water, &c.

CHAP.

CHAP. XI.

Evolution, *or Extracting the* Roots *out of all* Single Powers; *by one General Method.*

Sect. 1.

Evolution is the *Unravelling*, or as it were the *Unfolding* and *Reſolving* any propoſed *Power* or *Number*, into the ſame parts of which it was compoſed, or ſuppoſed to be made up. Now in order to perform that, it will be convenient to conſider how thoſe *Powers* are Compoſed, *&c.*

A *Square Number* is that which is *Equally Equal*; Or which is contained under two *Equal Numbers*. Euclid 7. Def. 18.

Thus the *Square Number* 4 is compoſed of the Two *Equal Numbers* 2 and 2. *viz.* 2×2=4.

Or the *Square Number* 9 is compoſed of the Two *Equal Numbers* 3 and 3. *viz.* 3×3=9. *According to Euclid.*

That is, if any *Number* be *Multiplied* into it ſelf; that Product is called a *Square Number*.

A *Cube* is that *Number*, which is *Equally Equally Equal*, or which is contained under *Three Equal Numbers*. *Euc.* 7. *Def.* 19.

Thus the *Cube Number* 8 is compoſed of the Three *Equal Numbers* 2 and 2 and 2. *viz.* 2×2×2=8, *&c.*

That is, if any *Number* be *Multiplied* into it ſelf, and that *Product* be *Multiplied* with the ſame *Number*; the ſecond *Product* is called a *Cube Number*.

Theſe *Two*, viz. the *Square*, and *Cube Numbers* borrow their *Names* from *Geometrical Extenſions* or *Figures*; as from the Three *Signal Quantities* mentioned in *Page* 2.

That is, a *Root* is repreſented by a **Line** or **Side**, having but one *Dimenſion*, *viz.* that of **Length** only.

The *Square* is a *Plain* or *Figure* of Two *Dimenſions*, having *Equal* **Length** and **Breadth**. The *Cube* is a *Solid Body* of Three *Dimenſions*; having *Equal* **Length**, **Breadth**, and **Thickneſs**: But beyond theſe Three, *Nature* proceeds not, as to *Local Extenſion*. That is, the *Nature* of Place or Space, admits no room for other ways of *Extenſion*, than *Length*, *Breadth*, and *Thickneſs*. Neither is it poſſible to form, or compoſe any *Figure* or *Body* beyond that of a *Solid*.

And therefore all the *Superior Powers* above the *Cube* or *Third Power*; As the *Biquadrat* or *Fourth Power*, the *Surſolid* or *Fifth Power*, &c. are beſt explain'd and underſtood by a *Rank* or *Series* of *Numbers* in *Geometrical Proportion*.

For Inſtance:

Suppoſe any *Rank* of *Geometrical Proportionals*, whoſe *Firſt Term* and *Ratio* are the ſame; And to them let there be

assigned a *Series* of *Numbers* in *Arithmetical Progression*, beginning with an *Unit* or 1, whose common *difference* is also 1, as in *Page* 79.

Thus, { 1 . 2 . 3 . 4 . 5 . 6 . 7 *Indices.*
2 . 4 . 8 . 16 . 32 . 64 . 128 &c. in ∺

Then are those *Numbers* in ∺ produced by a continued *Multiplication* of the *First Term* or *Root* into it self; And those in *Arithmetical Progression* or **Indices**, do shew what *Degree* or *Power* each *Term* in the *Geometrical Proportion* is of.

For *Example*; In this *Series* of ∺ 2 is both the *First Term* or *Root*, and common *Ratio* of the *Series*.

Then 2×2=4 the *Second Term* or *Square*.

And 2×2×2=8 Or 4×2=8 the *Cube* Or *Third Term*.

Again 2×2×2×2=16 Or 8×2=16 the *Fourth Term* or *Biquadrat*, And so on for the rest.

Note, *This is called* **Involution**, *viz. When any* Number *is drawn into it self, and afterwards into that* Product, &c. *it is said to be so often* involved *into it self*; *And the* Indices *are the* Exponents *of their respective* Powers *so* involved.

And according to these *Involutions*, is formed the following *Table* of *Powers*; wherein the *Root* is only one *Single Figure*.

Root, or single Side.	Square or 2d. Power	Cube, or the third Power	Biquadrate, or Square Squared; being the fourth Power.	Surfolid, or the fifth Power.	Square Cubed, or Cube Squared; the sixth Power	The second Surfolid, or Seventh Power	The Biquadrat Squared, or the eighth Power.	The Cube Cubed, or the ninth Power, &c
	Index (2)	Index (3)	Index (4)	Index (5)	Index (6)	Index (7)	Index (8)	Index (9)
1	1	1	1	1	1	1	1	1
2	4	8	16	32	64	128	256	512
3	9	27	81	243	729	2187	6561	19683
4	16	64	256	1024	4096	16384	65536	262144
5	25	125	625	3125	15625	78125	390625	1953125
6	36	216	1296	7776	46656	279936	1679616	10077696
7	49	343	2401	16807	117649	823543	5764801	40353607
8	64	512	4096	32768	262144	2097152	16777216	134217728
9	81	729	6561	59049	531441	4782969	43046721	387420489

This *Table* plainly shews (by *Inspection*) any *Power* (under the *Tenth*) of all the *Nine Figures*; and from thence may be taken

ken the neareſt *Root* of any *Square*, *Cube*, *Biquadrat*, &c. of y *Number* whoſe *Root* or *Side* is a *ſingle Figure*.

But if the *Root* conſiſt of *Two*, *Three*, or more places of *igures*, then it muſt be found by piece-meal, or *Figure* after *igure*, at ſeveral Operations.

The *Extraction* of all *Roots*, above the *Square* (*viz.* of the *ube*, *Biquadrat*, *Surſolid*, &c.) hath heretofore been a very dious and troubleſome Piece of Work: All which is now very uch ſhortned, and rendred *Eaſy*, as will appear further on.

When any *Number* is propoſed to have its *Root Extracted*, the *rſt* Work is to prepare it, by Points ſet over (or under) their roper *Figures*; according as the given *Power*, whoſe *Root* is ught doth require; And that's done by conſidering the *Index* the given *Power*, which for the *Square* is 2. for the *Cube* 3. r the *Biquadrat* is 4. &c. (as in the precedent *Table*). Then low ſo many Places of *Figures* in the given *Power*, for each *ngle Figure* of the *Root*, as its *Index denotes*; always beginning oſe *Points* over the *Place* of *Unity*, and *aſcend* towards the *Left Hand* if the given *Number* be *Integers*, and *deſcend* towards the *Right Hand* in *Decimal Parts*. As in theſe following.

Suppoſe any given *Number*; As 75640387246 which I ſhall ll along hereafter call the *Reſolvend*.

Then if it be *Required* to *Extract* any of the following *Roots*, muſt be pointed (according to the fore-mentioned Conſideration) this Manner:

Viz. For the
- *Square Root* Thus 75640387246
- *Cube Root* 75640387246
- *Biquadrat Root* 75640387246
- *Surſolid Root* 75640387246

Or ſuppoſe the *Number* to be 0,674035982

Then for the
- *Square Root* Thus 0,6740359820
- *Cube Root* 0,674035982
- *Biquadrat Root* 0,674035982000

Now the Reaſon of *Pointing* the given *Reſolvend* in this anner; *viz.* the allowing *Two Figures* in the *Square*; *Three igures* in the *Cube*, and *Four Figures* in the *Biquadrat*, &c. For ne *Figure* in the *Root*, may be made Evident ſeveral ways; but

I think it's eaſily conceived from the *Table* of *ſingle Powe* wherein you may obſerve that all the *Powers* of the *Figure* (which is but a *ſingle Figure)* have the ſame *Number* of Places *Figures*, as the *Index* of thoſe *Powers denotes*: Therefore many Places of *Figures* muſt be taken or aſſigned for every *ſing Figure* in the *Root*. Conſequently by theſe *Points* is know how many Places of *Figures* there will be in the *Root*, *viz.* many *Points* as there are, ſo many *Figures* there muſt be in th *Root*, and whether they muſt be *Integers* or *Decimal Parts*, eaſily determined by the reſpective Places of the *Points*.

Sect. 2. *To* Extract *the* Square Root.

And *Firſt* how to *Extract* the *Square Root*, according to th common *Method*.

Having *pointed* the given *Reſolvend* into *Periods* of *T Figures*, as before directed; then by the *Table* of *Powers* (o *otherwiſe)* find the greateſt *Square* that is contained in the *fir Period* towards the *Left Hand*; (ſetting down its *Root*, like *Quotient Figure* in *Diviſion)* and *Subſtract* that *Square* out o the ſaid *Period* of the *Reſolvend*: To the *Remainder* brin down the next *Period* of *Figures*, for a *Dividend*, and doubl the *Root* of the *firſt Square* for a *Diviſor*; inquiring how oft i may be had in that *Dividend*; So as when the *Quotient Figur* is *annexed* to the *Diviſor*, and that *increaſed Diviſor* being *Multiplied* with the ſame *Quotient Figure*, the *Product* ma be the greateſt *Number* that can be taken out of that *Dividend* which *Subſtract* from the ſaid *Dividend*, and to the *Remainder* bring down the next Period of *Figures*, for another New *Di vidend*: Then ſee how often the Laſt *increaſed Diviſor*, can be had in the New *Dividend*; (*with the ſame Caution as before viz.*) So as that the *Quotient Figure* being *Annexed* to the *Diviſor*, and that *increaſed Diviſor Multiplied* with the ſame *Quotient Figure*, their *Product* may be the greateſt *Number* that can be *Subſtracted* from the New *Dividend*. (As before). And ſo proceed on from *Period* to *Period*; (viz. from *Point* to *Point)* in the very ſame *manner*, until all be finiſhed.

An *Example* or *Two* being well *Obſerved* will render the Work of forming the New *Diviſors*, &c. more *Plain* and *Eaſy*, than can be Expreſſed in a *Multitude* of Words.

Example 1. Let it be required to *Extract* the *Square Root* out of 572199960721. This *Reſolvend* being prepared or pointed as before directed, will ſtand

Thus,

Thus, 572199960721 (756439 the *Root*.
49=the greateſt *Square* in 57.

Diviſor 145)	821	
5	725=145 × 5	
Diviſor 1506)	9699	
6	9036=1506 × 6	
Diviſor 15124)	66396	
4	60496=15124 × 4	
Diviſor 151283)	590007	
3	453849=151283 × 3	
Diviſor 1512869)	13615821	
9	13615821=1512869 × 9	

Proof 756439×756439 = 572199960721 the *Reſolvend.*

Example 2. What's the *Square Root* of 1850701,764025?

Operation 1850701,764025 (1360,405

1	
23)	85
3	69
266)	1607
6	1596
17204)	1101,76
4	1088 16
1720805)	13 604025
5	13 604025
	(0)

{ Hence 1360,405 is the *Root required.*

Ex. 3. What's the *Square Root* of 0,06076225 *Decimal Parts*?

Operation 0,06076225 (0,2465 the *Root* required.

	,04=,2 × 2
,44)	207
4	176
,486	3162
6	2916
,4925)	24625
5	24625
	(0)

Proof { 0,2465 × 0,2465 = 0,06076225 the *Reſolvend.*

What

What is here done in Whole *Numbers*, Mix'd *Numbers* and *Decimals*, may alſo be done in *Vulgar Fractions*; if you firſt Change the given *Fraction* into *Decimals*; (As in *Sect.* 5. *p.* 68.

Example 4. Let it be required to Extract the *Square Root* of $\frac{16}{25}$. *Firſt* $\frac{16}{25}=0{,}64$

Then 0,64 (,8 *the* Root *required.*
,64
(0)

In theſe *Four Examples* the *Reſolvend* hath been a *perfect Square*; and therefore the *Root* hath been Extracted without Leaving any *Remainder*: But it very often happens that the *Reſolvend* is not a true *Figurate Number*, according to the propoſed *Power*. That is, it's not a *perfect Square*, *Cube*, *Biquadrat*, &c. and then ſomething will *Remain* after the Extraction hath been made throughout all the *Points*. Such *Numbers* are called Surd *Numbers*, and their *Roots* can never be truly found, but will become a *Continued Series*, *ad infinitum*. If to the *Remainder* there be ſtill *Annexed Cyphers* according to the propoſed *Power* requires, *viz.* by *Two*'s in the *Square*, *Three*'s in the *Cube*, *Four*'s in the *Biquadrat*, &c. And the *Operations* continued on as before.

Example 5. Suppoſe it were required to Extract the *Square Root* of 6968.

Operation 6968 (83,4745, &c.
64
163) 568
3 489
1664) 79,00
4 66 56
16687) 12 4400
7 11 6809
166944) 759100
4 667776
1669485) 9132400
5 8347425
1669490) 784975 &c.

Thus the *Root* of any *Surd Number* may be *continued* on to what *Exactneſs* you pleaſe, but cannot be truly found.

In my *Compendium of Algebra*, Chap. 9. I have propoſed another way of *Extracting* the *Square Root*, and there given *Examples* of the Work: which to avoid Prolixity is thus;

Havi

Having *pointed* the given *Reſolvend*, and taken the *greateſt Square* to the *firſt Point* from it, as before. Then *Divide* the *Remainder* of the whole *Reſolvend* by 2 (that is, *halve* it) and *Point* it a New. (This I call a *New Dividend*) Then make the *Root* of the *firſt Square* a *Diviſor*, *inquiring* how oft it may be found in the New *Dividend* to the next *Figure* forward, reſerving that *Figure* under the next *Point*, for the *half Square* of the *Quotient Figure*. Which being found, *Multiply* the *Diviſor* with it, *adding* to that *Product* the *Tens* of the *half Square* if there be any; As in plain *Diviſion*.

Then *annex* the *Quotient Figure* to the laſt *Diviſor* for a new *Diviſor*, with which proceed in all Reſpects as with the laſt *Diviſor*; and ſo on until all be finiſhed.

Example 6. What's the *Square Root* of 2990667969

. . . . .

Operation 2990667969

— 25 (5 The *firſt Single Root*

2) 490667969 The *Remainder* to be *Divided* by 2.

. . . . .

firſt Root 5) 24533398 4,5 (54687

+ 4 208 = 5×4 : + ½ the *Square* of 4 *viz.* $\frac{16}{2}=8$

Diviſor 54) 3733

+ 6) 3258 = 56×6 : + ½ the *Square* of 6.

Diviſor 546) 47539

+ 8) 43712 = 546×8 : + ½ the *Square* of 8.

Diviſor 5468) 382784,5

+ 7 382784,5 = 5468×7 : + ½ the *Square* of 7.

(0)

Hence the *Root* is found to be 54687 *As was required.*

All the *difficulty* in this *Method* is only in the true placing of the *half Square* of the *Quotient Figure*, when in happens to be an *Odd Number*; In that *Caſe* you muſt bring down one *Figure* more of the *Dividend*; viz. of the next *Period*; *under* which, place the odd 5 that will *always ariſe* from the *half Square* of an *odd Number*: As 7 whoſe *Square* is 49; the half of which is 24,5 to be placed as in the laſt *Operation* of this *Example*.

N. B. *When the* Number *of* Figures *in the* Root *of any* Surd Number *are limited*; *you need not proceed in* Extracting *the whole* Root *as before*; *but only to One* Figure *more than half the deſigned* Number *of* Figures; *for the reſt may be obtained by* Plain Diviſion *only*.

Example 7. Suppose it were required to *Extract* the *Squa* *Root* of 7 (a *Surd Number*) to have 12 places of *Figures* in it.

```
                 .
                 7  (2,645751     First part of the Root.
                 4
               ----
Remainder        3 .
   2)            1,50=half the Remainder.
 + ,6            1,38=2×,6 : + ½ the Square of 0,6=0,18
 ----            ----.
  2,6)           1200
 + ,04           1048
 -----           -----.
  2,64)          152000
 + ,005          132125
 ------          ------.
  2,645)         1987500
 + ,0007         1851745
 -------         -------.
  2,6457)        13575500
 + ,00005        13228625
 --------        --------.
  2,64575)       34687500
 + ,000001       26457505
 --------       ---------
  2,645751        8229995
```

Having thus got 7 of the 12 *Figures* required in the *Root* the rest may be *easily found* by the *contracted* way of *Division* *proposed in Page* 68.

```
Thus, 2,645751)  8229995 | (2,64575131106
                 7937253 |
                  292742 |
                  264575 |
                   28167 |
                   26457 |
                    1710 |
                    1097 |
                    (13) |
```

Hence I find the *Root* of 7 to be 2,6457131106 *As wa* *required.*

Thus you have *Two* ways of *Extracting* the *Square Root* either of them may be *practiced* as every one likes best.

Sect

Sect. 3. *To* Extract the Cube Root.

The *Method* that I ſhall here propoſe for *Extracting* the *Cube Root* admits of Two *Caſes*; both which are to be very well obſerved.

Having *pointed* the *given Reſolvend*, (as before directed) *viz.* into *Periods* of *Three Figurei*; Then ſeek a *Cube Number* by the *Table* of *Powers* (or otherwiſe) that comes the neareſt to the *firſt Period* of the *Reſolvend*, whether it be *Greater* or *Leſs* than that *Period*.

Caſe 1. If the *Cube Number* ſo taken, be *Leſs* than the *Firſt Period* of the *Reſolvend*,

Call its Root Leſs *than* Juſt.

And *Subſtract* that *Cube* from the *Firſt Period* of the *Reſolvend*.

Caſe 2. But if that *Cube* be greater than the *Firſt Period* of the *Reſolvend*,

Call its Root More *than* Juſt.

And *Subſtract* the *Reſolvend* from that *Cube*, *Annexing Cyphers* to it, that ſo *Subſtraction* may be made.

To the firſt *Root* whether it be *Leſs*, or *More* than *Juſt*, *Annex* ſo many *Cyphers* as there are remaining *Points* over the whole *Numbers* of the *Reſolvend*, and *Multiply* it with 3; Then make that *Product* a *Diviſor*; by which you muſt *divide* the *Difference* between the *Reſolvend* and the foreſaid *Cube*, then will that *Quotient* be the *Reſolvend Depreſſed* to a *Square*; and therefore it muſt be *pointed* as ſuch: *viz.* into *Periods* of *Two Figures* each. That being done, make the *Firſt Root* (without thoſe *Cyphers* that were *Annex'd* to it) a *Diviſor*, inquiring how oft it may be found in the *Firſt Period* of the *New Reſolvend*, (as before in *Extracting* the *Square Root*) with this Conſideration, that if the *Root*, (now a *Diviſor*) be *Leſs* than *Juſt*, as in *Caſe* 1. you muſt Annex the *Quotient Figure* to it, and then *Multiply* the *Root* ſo *increaſed*, into the ſaid *Quotient Figure*; Setting down the *Units* place of their *Product* under the *Pointed Figure* of that *Period*, *Subſtracting* it, as in *Diviſion*. And ſo on from one *Period* to another, As before.

But if the ſaid *Root* (now a *Diviſor*) be *More* than *Juſt*, as in *Caſe* 2. Then you muſt *Subſtract* the *Quotient Figures* from a *Cypher Annex'd*, or ſuppoſed to be *Annexed* to the ſaid *Diviſor*; *Multiplying* the *Root* ſo *Decreaſed* into the *Quotient Figure*; Setting down their *Product* as before, *&c.* An *Example* or *Two* in each *Caſe* will render the Work *Plain* and *Eaſy*.

Example 1.

What's the *Cube Root* of 146363183 the given *Reſolvend*, to be pointed thus 146363183 (5 the *Firſt Root*, *Leſs* than *Juſt*.
125=the neareſt *Cube* to 146

500 × 3=1500) 21363183 (14242,12 New *Reſolvend*.
Firſt *Root* 5) 14242,12 (527 the *Root* required.
+ 2 104
1. *Diviſor* 52) 3842
+ 7 3689
2. *Diviſor* 527 (153) the *Remainder* to be rejected.

Here the *Root* 527 is the *true Root* at the *firſt Operation*, as may be *eaſily* tried by *Involving* it.

That is 527 × 527 × 527 = 146363183 the given *Reſolvend*. But if it had not been the *true Root*; Then every thing that hath been here done muſt have been repeated; *Only inſtead* of the *firſt ſingle Root* (*viz.* 5) you muſt have taken the *Increaſed Root* (viz. 527) and this I call a *Second Operation*; Which would *Increaſe* the *Laſt Root* to *Nine* places of *Figures*; viz. every *Operation Triples* the *Number* of places in the *Laſt Root*; As will appear further on.

N. B. *It often happens that Four, or ſometimes Five Places of Figures may be taken into the* Root; *Eſpecially when the Second Place proves to be a Cypher. That is, when the Firſt Cube comes very near to the Firſt Period of the* Reſolvend.

Example 2.

What's the *Cube Root* of 67507824239 (4000 *Root Leſs*
Firſt neareſt *Cube*=64 (than *Juſt*.

Root 4000 × 3=12000) 3507824239 (292318,68
4) . . . .
+ 0 292318,68 (4071,8
1. *Diviſor* 40) 2923
+ 7 2849
2. *Diviſor* 407) 7418
+ 1 4071
3. *Diviſor* 4071) 3347,68
+ 8 3257,44 &c.
Root = 4071,8

In this *Example* I have taken *Five Figures* into the *Root*, becauſe the *Second Place* proved to be a *Cypher*. And in theſe *Five*

ve the *Exceſs* is not an *Unit* in the *Laſt Place*; for if there re made a *Second Operation*, the *Root* would be 4071,78 *&c.* s may be eaſily tried.

Example 3.

Let it be required to *Extract* the *Cube Root* out of this umber; . . . . . .

Viz. 976379602989073960279630298890

he neareſt *Cube to* 976 is 1000 *whoſe Root is* 10 *being* More an Juſt . . . . . .

s *Cubes* is 1000000000000000000000000000000

\+ 976379602989073960279630298890 the *Reſolvend.*

Remains 23620397010926039720369701110

he Firſt *Root* 10000000000 × 3=30000000000 the *Diviſor* hen 30000000000) 23620397010926039720369701110 (*

. 000000000

irſt *Root* 10) *7873465670308679990 = *New Reſolv.*

— 00 . . . . (007929

Diviſor 100 7873

— 7 6951

Diviſor 993 92246 Firſt *Root*=10000000000

— 9 89289 — 007929

Diviſor 9921 295756 9920710000

— 2 198416

Diviſor 99208 9734070

— 9 8928639

Diviſor 992071 &c.

At this *Firſt Operation* I take but 99207 to which I Annex 5 yphers for the *Remaining Points*, viz. 9920700000 which being volved to the *Third Power* or *Cube*, for a *Second Operation*, will be 9763981560227430000000000000000

— 976379602989073960279630298890 *Reſolvend*

Remains 18553033669039720369701110

he Laſt *Root* 9920700000 × 3=29762100000 the *new Diviſor* hen 29762100000) 18553033669039720369701110 (*

* 623377841921091 The *Quoticnt* or *New Reſolvend.*

Laſt *Root* 99207 being *More* than *Juſt*, therefore the new uotients muſt be *Subſtracted*, as in the *Laſt Operation.*

Thus

Thus

99207)	623377841	921091	(62836,45
— 6	5952384		Note *in th*
Diviſor 992064)	28139441		*Operation*
— 2	19841276		*the Figur*
Diviſor 9920638	8298165	92	*on this ſi*
— 8	7936509	76	*the Line*
(*) 99206372	361656	1610	*all the ne*
— 3	297619	1151	Diviſors
992063717	64037	045991	*ter the*
— 6	59523	822984	*are uſele*
9920637164	4513	22300700	*and migh*
— ,4	3968	25486544	*have been*
99206371636	544	9681415600	*mitted.*
— ,05	496	0318581775	
9920637163,55		&c.	

Laſt Root 9920700000
— 62836,45 &c.

9920637163,55 the *Root required.*

Thus I have obtained the *Cube Root* to *Twelve Places* of *Figures, viz.* 9920637163,55 at *Two Operations*; being but an *Unit* too much in the *Laſt Place* of it, as may be tried by *involving* it to a *Cube*, and comparing that *Cube* with the given *Reſolvend.*

In the ſame manner the *Cube Roots* of *Decimal Parts*; or of *Vulgar Fractions*, being firſt *changed* into *Decimals*, may be *Extracted.*

Sect. 4. *To* Extract *the* Biquadrat Root.

In *Extracting* the *Biquadrat Root*, or that of the Fourth *Power*; (and indeed the *Roots* of all even *Powers*) there is ſome ſmall *Difficulties*, not ſo eaſily *Expreſs'd* and *Explain'd* in a few Words, as they are by an *Algebraick Theorem* (ſuch as ſhall be ſhewed further on) I have therefore in this place, made choice of *Extracting* ſuch *Roots* by *Two* ſeveral *Extractions*; And the rather, becauſe *I preſume* the *Reader* by this time thoroughly acquainted with the Buſineſs of *Extracting* the *Square Root*, by which this may eaſily be performed. Thus:

Firſt *Extract* the *Square Root*, of the propoſed *Reſolvend*, Then the *Square Root* of that firſt *Root* will be the *Biquadrat Root* required.

Example

Example 1. What's the *Biquadrat Root* of 4857532416 ? *irst Extract* its *Square Root*,

```
        . . . . .
Thus    4857532416
   —    36=the greatest Square, whose Root is 6.
        ----------
        1257532416 Remainder to be Divided by 2.

                  . . . .
irst Root 6 )     628766208 ( 69696
     +  9         5805
        ---       ----
        69         4826
         6         4158
        ---       ----
        696        668620
     +    9        626805
        ----       ------
        6969        418158
                    418158
                    ------
                     (0)
        . . .  { being the first Root, whose Square Root
Then    69696  {   must now be Extracted.
     — 4
        -----
        29696  Remainder to be Divided by 2.

                 . .
irst Root 2)     14848 ( 264 the Biquadrat Root as was requir'd.
     + 6         138
        ---      ----
        26)      1048
         4       1048
        ---      ----
        264      (0)
```

his is so Easy I need not *insert* any more *Examples*.

Sect. 5. *To* Extract *the* Sursolid Root.

Having *pointed* the given *Resolvend* according as its *Index enotes*; *viz.* into *Periods* of *Five Figures*; Seeking such a *ursolid Number* in the *Table* of *Powers* (or otherwise) as comes e nearest to the *First Period* of the *Resolvend*, whether *Greater* r *Less*; and call its respective *Root* accordingly; *viz. More* han *Just*; Or *Less* than *Just*; *Annexing* so many *Cyphers* to , as there are remaining *Periods* of whole *Numbers* in the *esolvend*. As before in *Extracting* the *Cube Root*.

Then find the *Difference* between the *Resolvend*, and the *Sursolid Number* so taken, by *substracting* the *Lesser* from the *Greater*, as before in the *Cube*). Next find the *Cube* of the aforesaid *Sursolid*

Surſolid Root with its *Annexed Cyphers*, (which you may alſo by the *Table* of *Powers*) and *Multiply* that *Cube* with 5 th *Index* of the *Surſolid*, the *Product* muſt be a *Diviſor*, by whic the *Difference* between the *Reſolvend* and the *Surſolid* Numbe muſt be *Divided*; that ſo it may be *depreſſed* to a *Square* (a before in the *Cube*) which muſt be pointed into *Periods* of Tw *Figures* each, calling it the New *Reſolvend*, (as before). The make the *Firſt Root*, without its *Cyphers*, a *Diviſor*, *Inquirin* how oft it may be found in the *Firſt Period* of the New *Reſol vend*, with this *Conſideration*, if the *Root* (now a *Diviſor*) b *Leſs* than *Juſt*, you muſt *Annex Twice* the *Quotient Figure* t it; but if it be *More* than *Juſt*, you muſt *Subſtract Twice* th *Quotient Figure* from a *Cypher* either *Annexed*, or ſuppos'd t be *Annexed* to that *Diviſor* or *Root*, *Multiplying* it ſo *Increaſe* or *Diminiſhed*, with the ſaid *Quotient Figure*, ſetting down their *Product*, &c. as before. An *Example* in each *Caſe* wil render it plain and eaſy.

Example 1. Suppoſe it be required to *Extract* the *Surſolid Root* out of this *Number* 12309502009375.

12309502009375 The *Reſolvend Pointed.*

The neareſt *Surſolid Number* to 1230, the Firſt *Period* of the *Reſolvend*, is 1024, whoſe *Root* is 4 being *Leſs* than *Juſt*.

Therefore 12309502009375
— 1024

2069502009375 their *Difference.*

Next the *Cube* of 400 is 64000000 *per Table*, &c.
And 64000000 × 5 = 320000000 the *Diviſor.*
Then 320000000) 2069502009375 (6497 &c.

First *Root*	4)	6467	(15
+ 1 × 2 = 2		42	*First* Root = 400
			+ 15
Diviſor	42	2267	
+ 5 × 2 = 10		2150	the true *Root* 415 as *required.*

That is 415 is the *Surſolid* Root of the given *Reſolvend.* As may be *eaſily* tried by *involving* it to the *Fifth Power.*

Viz. 415×415×415×415×415=12309502009375 the given *Reſolvend.*

Example

Example 2. What's the *Surſolid Root* of 232783455987 3
The *neareſt Surſolid Number* to 232 is 243 whoſe *Root* is 3
being *More* than *Juſt*.

Therefore 233000000000
— 242783455987

Remains 10216544012 For a *Dividend*.

The *Cube* of 300 is 27000000 And 27000000×5=135000000
Then 135000000) 10216544012 (756,7810 New *Reſolvend*.

Firſt Root	300)	756,7810 (2,558
— 02 × 2 =	4	592
Diviſor	296)	164,78
— ,5 × 2 =	1,0	147,50
2. *Diviſor*	295,0)	17,2810
— ,05 × 2 =	,10	14,7450
	294,90)	2,53600 &c.

The *Firſt Root* was 300, being *More* than *Juſt*.
Therefore it is — 02,558

The *New Root*, 297,442 And is very near the true *Root*, which is 297,436 &c. Now the *Reaſon* why this *Root* comes out to ſo many places of *Figures* at the Firſt *Operation*; is becauſe the *Firſt Surſolid Number* was ſo near the *Reſolvend*, &c. As before.

Sect. 6. *To* Extract *the* Root *of the* Square Cubed.

This may be eaſily performed by *Two Extractions*, according as its Name denotes.

Thus, Firſt *Extract* the *Square Root* of the given *Reſolvend*; then *Extract* the *Cube Root* of that *Square Root*: And it will be the *Root Required*. That is, it will be the *Root* of the *Sixth Power*.

Or thus, Firſt *Extract* the *Cube Root* of the *Reſolvend*, then *Extract* the *Square Root* of that *Cube Root*: And it will be the *Root Required*.

Example 1. Let it be Required to *Extract* the *Square Cubed Root* out of this *Number* 1452205373535 15625 the *Reſolvend*.

Firſt I *Extract* the *Square Root* of this *Reſolvend*, which I like to be the beſt and eaſieſt way.

Thus . . . . . .
145220537353515625
— 9

Remains 552205373535156 25 To be halved.
. . . . . .

Then 3)	2761026867675781 2,5	(381078125
+ 8	272	
38)	4102	
+ 10	3805	
3810)	2976867	
+ 7	2667245	
38107)	3096226	
+ 8	3048592	
381078)	47634757	
+ 1	38107805	
3810781)	95269528	
+ 2	76215622	
38107812)	190539061 2,5	
+ 5	190539061 2,5	
381078125	(0)	

Having found the *Square Root* of the given *Reſolvend*, I proceed to *Extract* the *Cube Root* of that *Square Root*.

That is, of 381078125
— 343 = the neareſt *Cube*, its *Root* is 700

Then 700 × 3 = 2100) 38078125 (18161
. .

Firſt Root	7)	18161	(25
	+ 2	144	
1. *Diviſor*	72)	3761	
	+ 5	3625	
2. *Diviſor*	725	(136)	

Firſt *Root* 700
+ 25
725

Hence I find 725 to be the *Square Cube Root* required; and may eaſily be tried by *Involving* it to the *Sixth Power*.

That is, 725×725×725×725×725×725 will be found = 145220537353515625 the given *Reſolvend*.

Sect.

Sect 7. *To* Extract *the* Root *of the* Seventh Power.

Having *pointed* the given *Reſolvend*, as its *Index Denotes, viz.* into *Periods* of *Seven Figures*, ſeek out ſuch a *Number* of the *Seventh Power*, by the *Table* of *Powers*, as comes neareſt to the *Firſt Period* of the *Reſolvend*; whether it be *Greater* or *Leſſer*, calling its reſpective *Root More* than *Juſt*,, or *Leſs* than *Juſt*, *Annexing* its proper *Number* of *Cyphers*, &c. As in the *Cube* and *Surſolid*.

Then find the *Difference* between the given *Reſolvend*, and that *Number* of the *Seventh Power* (found by the *Table* of *Powers*) by *Subſtracting* the *Leſſer* from the *Greater*.

Next find the *Surſolid* or *Fifth Power* of that *Root* with its *Annexed Cyphers* (which you may alſo do by the *Table* of *Powers*) and *Multiply* that *Surſolid Number* with 7, the *Index* of the given *Reſolvend*, that *Product* muſt be a *Diviſor*, by which the foreſaid *Difference* muſt be *Divided*; that ſo it may be *Depreſſed* to a *Square*, to be *Pointed*, &c. as before in the *Cube*, &c. then make the *Firſt Root*, without its *Cyphers*, 'a *Diviſor*; working with it and the New *Reſolvend* (as before) only here you muſt *Increaſe*, or *Diminiſh* the *Diviſor* with *Thrice* the *Quotient Figure*.

Example.

What's the *Second Surſolid Root*, or that of the *Seventh Power*.

of 37323655395507812 5 the *Reſolvend pointed*.
— 2178 *the neareſt Number of the Seventh Power*.

15543655395507812 5 their *Difference*.

The *Firſt Root* is 300 being *Leſs* than *Juſt*; And the *Fifth Power* of 300 is 243000000000 which being *Multiplied* with 7 is 1701000000000 for a *Diviſor*, by which the aforeſaid *Difference* muſt be *Divided*; which *Contracted* may ſtand thus 1701) 15543655 (9137,95 &c.

Firſt Root 3)	9137	(25	
+ 2 × 3 = 6	72		*Firſt Root* = 300
1. *Diviſor* 36	1937		+ 25
+ 5 × 3 = 15	1875		*True Root* 325
1. *Diviſor* 375	(62)	the *Remainder* to be rejected as before.	

Hence I have found 325 to be the *True Root* required, that is, the *True Root* of the *Seventh Power*.

I think it needleſs to proceed farther; *viz.* to inſert *Examples* of *higher Powers*. For if what is already done be well underſtood, it will be *Eaſy* to *conceive* how to proceed in *Extracting* the *Root* of any *Single Power* how *high* ſoever it be (for the *Method* is *General* and alike in all *Powers*) due regard being had to their *Indices*; and to the *Firſt Single Side* or *Root*. That is, whether it be *More*, or *Leſs* than *Juſt*, &c.

Yet methinks I hear the young *Learner* ſay, 'tis *poſſible* to follow the *Directions* and *Examples*, as they are here laid down; but ſtill here is not the Reaſon why they are ſo, and ſo *performed*; And why there ſhould be a *Remainder Left* after the *True Root* is found; *viz.* when the given *Reſolvend* hath a *True Root* of its kind.

'Tis true, the *Reaſons* of theſe are not here laid down; neither *indeed* can they be rendred ſo plain and intelligible by Words, as by an *Algebraick Proceſs*, from whence the *Theorems* or *Rules* here given, had their *firſt Invention*; as ſhall be ſhewed in the *next Part*, when I come to treat of *Reſolving Compounded* or *Adfected Æquations*; however, take this ſhort and *general* Account of this *Method*.

This, and all other of the new *Methods* of *Converging Series* (as they are called) are very *different* from the *former* (and ſtill *common*) *Methods* of *Extracting Roots*, which requires the *Firſt ſingle Side* or *Root* of the *Firſt Period* (in any *Reſolvend*) to be taken *exactly True*, and then by *Involving*, and other *Tedious* Ways of ordering it, there is formed a *Diviſor*; which helps to *grope* out by *Trials* a *Second Figure* in the *Root*. And ſo *proceeds* on from *Point* to *Point*; ſtill repeating the whole Work for every *ſingle Figure* that comes into the *Root*. And if by *Chance* there be a Miſtake or *Error* committed in any one *Figure* (as 'tis poſſible there may) it ſpoils the whole *Proceſs*, which muſt then be wholly begun a *New*, or at *Leaſt* from that *Part* of it where the *Error firſt entered*.

But the *Nature* and *Deſign* of the *Method* which I have here *Laid down* is quite otherwiſe; it being ſo *contrived*, as to *gradually Leſſen* the *Difference* betwixt any propoſed *Power*, and the like *Power* of another *Number* aſſumed; *viz.* it *Leſſens* that *Difference* until it's either *quite Vanquiſhed*, or become ſo *Infinitely ſmall* as to be *Inſignificant*.

Therefore when any *Number* is propoſed to have its *Root Extracted*; it is here required to take the next *neareſt Root* of the *firſt Period* in the *Reſolvend*; that ſo the *Difference* betwixt

twixt the given *Reſolvend*, and the *Homogeneal Power* (viz. e *like Power)* of the *Root* thus taken, may be *Leſs* either *Exceſs*, or *Defect*. Which *Difference* being *Reduced*, or *Depreſſed Lower*, becomes ſo prepared, that by plain *Diviſion* (*comparatively*) there will ariſe ſuch *Quotient Figures* as will both *Correct* and *Increaſe* the *Firſt Root* to *Three places* of *Figures* at *leaſt*, ſometimes to *Four*, or *Five places* of *Figures*; according as the ſaid *Firſt difference* happens to be *More* or *Leſs*; of which you may have obſerved Inſtances): But yet there will be a *Remainder* left, and *perhaps* an *Exceſs* or *Defect* in the *Root* ſo *Increaſed*, viz. in the *Laſt Figure* of it.

Now to *rectify* the ſaid *Exceſs* or *Defect* in the *Root*, and to *diſcover* whether the given *Reſolvend* be a *true Figurate Number*, or not: That is, Whether it have a *True Root* of its kind. It will be *neceſſary* to make a *Second Operation*; by taking the *Root* ſo *Increaſed*, and *proceeding* with it and the given *Reſolvend*, in all reſpects as in the firſt Work (like to the *Third Example* of *Extracting* the *Cube Root*) I ſay, if the given *Reſolvend* have a *True Root*, it will appear at this *Second Operation*, and all the aforeſaid *Differences*, &c. will be *Vaniſh'd*; *Provided* the *Root* required is not to have more than *Three* (or *Four*) places of *Figures* in it.

But if the *Root* be to have more than *Three Figures* in it; Or, that the given *Reſolvend* prove to be a *Surd Number*. Then there will be a *Difference* as before; which will afford *Quotient Figures* to Rectify and Increaſe the *Root* laſt taken, to Three times as many places of *Figures*, as it had at the Beginning of that *Second Operation*. As you may ſee in the aforeſaid *Example* 3. of the *Cube Root*; wherein that *Root* is increaſed to Twelve places of *Figures* at *Two Operations*: which if it were to be *Extracted* the *Old* (and ſtill *Common*) way, it would require at leaſt Forty times the Number of Figures I have here uſed.

Again, If there chance to be a Miſtake committed in any *Operation* perform'd by the *Method* here laid down, that Miſtake will not deſtroy the precedent Work, but will be Rectified in the *next Operation*, although it were not diſcovered before. And thus you may proceed on to a *Third Operation*, which will afford 27 places of Figures in the *Root*, &c. with very *little* Trouble, if compared with former *Methods*.

This brief Account, which I have here given (*by way of Explaining the Nature of this Method of Extracting Roots*) being well conſidered and compared with the ſeveral *Operations* of the foregoing *Examples*, muſt needs help the *Learner* to form ſuch an Idea of it, that he cannot (I preſume) but underſtand how

how to proceed in *Extracting* the *Root* out of any *Single Powe* how high ſoever it be; without the Help of an *Algebrai* *Theorem*. Not, but when that comes to be once underſtood the Work will be much readier and eaſier perform'd: As wi appear in the *next Part*.

I did intend to have here inſerted, the whole Buſineſs *Intereſt* and *Annuities*; but finding that it would require to *large* a *Diſcourſe*, to ſhew the *Grounds* and *Reaſons* of the ſeveral *Theorems uſeful* therein, I have therefore reſerved that Work for the Cloſe of the *next Part*. Neither indeed can the Raiſing o thoſe *Theorems* be ſo well delivered in *Words*, as by an *Algebraick* way of *Arguing*; which renders them not only much *ſhorter* but alſo *plainer* and *eaſier* to be underſtood.

I have alſo *Omitted* that *Rule* in *Arithmetick*, uſually called the *Rule of Poſition*, or *Rule of Falſe*: Becauſe all ſuch *Queſtions* as can be Anſwered by that *Gueſſing Rule*, are much better done by any one who hath but a very ſmall ſmattering of *Algebra*. I ſhall therefore conclude this Part of *Numerical Arithmetick*; and proceed to that of *Algebraick Arithmetick*; wherein I would adviſe the young *Learner* not to be too haſty in paſſing from one *Rule* to another, and then he will find it very eaſily to be attained.

A N

AN INTRODUCTION TO THE Mathematicks.

PART II.

PROEM.

HAving formerly wrote a small Tract of **Algebra**, *perhaps it may seem (to some,) very improper to write again upon the same Subject; but only (as the usual Custom is) have referr'd my* Reader *to that Tract. However, because the following Parts of this Treatise are managed by an Algebraick Method of Arguing; which may fall into the Hands of those who have not seen that Tract, or any other of that Kind; I thought it convenient to accommodate the Young Geometer with the First Elements, or Principal Rules, by which all Operations in this Art are performed; that so he may not be at a Loss as he proceeds farther on: Besides, what I formerly wrote was only a Compendium of that which is here fully handled at large.*

The Principal Rules are **Addition, Substraction, Multiplication, Division, Involution,** and **Evolution,** as in common **Arithmetick** (but differently perform'd); and therefore some call it **Algebraick Arithmetick.** Others call it **Arithmetick in Species,** because all the *Quantities* concerned in any *Question, remain* in their *Substituted Letters* (howsoever manag'd by *Addition, Substraction, Multiplication,* &c.) without being *destroyed* or *changed* into others, as *Figures* in common *Arithmetick* are.

Mr. *Harriot* called it **Logistica Speciosa,** or *Specious Computation.*

CHAP.

CHAP. I.

Concerning the Method *of* Noting *down* Quantities; *and* Tracing *their* Steps, *&c.*

Sect. 1. *Of* Notation.

THE *Method* of *Noting* down *Letters* for *Quantities*, is *various*, according to every one's Fancy; But I shall here follow the same as in my former *Tract*: And *represent* the *Quantity* sought (be it *Line* or *Number*, *&c.*) by the small (*a*) and if more *Quantities* than *One* are sought, *represent* them by the other *small Vowels*, *e. u.* or *y*.

The given *Quantities* are *represented* by the *small Consonants* *b. c. d. f. g. &c.*

And for *Distinction* sake, *mark* the *Points* or *Ends* of *Lines* in all *Schemes*, with the *Capital* or Great *Letters*, *viz. A. B. C. D. &c.*

When any *Quantity* (either given or sought) is taken more than *Once*, you must *prefix* its *Number* to it; As 3*a* stands for *a* taken *Three times*, or *Three times a*, and 7*b* stands for *seven times b*, &c.

All *Numbers* thus *prefixt* to any *Quantity*, are called *Coefficients* or *Fellow-Factors*; because they *Multiply* the *Quantity*; And if any *Quantity* be without a *Coefficient*, it is always suppos'd or understood to have an *Unit* prefix'd to it; As *a* is 1*a*, or *b* is 1*b*, &c.

The *Signs* by which *Quantities* are *chiefly managed* are the same; and have the same *Signification*, with those in the *First Part*, *page* 5. which I here presume the *Reader* to be very well acquainted with. To them must be here *Added* these *Three* more;

Viz. { ⊖ / ɯ / √ } the *Sign* of { *Involution.* / *Evolution, or Extracting Roots.* / *Irrationality, or Sign of a* Surd Root. }

All *Quantities* that are express'd by *Numbers* only (as in *Vulgar Arithmetick*) are called *Absolute Numbers*.

Those *Quantities* that are *represented* by *single Letters*, as *a. b. c. d.* &c. or by several *Letters* that are immediately joined together; As *ab. cd.* or 7*bd.* &c. are called *Simple* or *Single* whole *Quantities*.

But when *different Quantities* represented by *different* or unlike *Letters*, are *connected* together by the *Signs* (+ or −); As *a*+*b*. *a*−*b*. or *ab*−*dc*. &c. they are called *Compound whole Quantities*.

And

And when *Quantities* are *Expreſs'd* or ſet down like *Vulgar ractions*, Thus $\frac{a}{b}$ Or $\frac{a+b}{d}$ Or $\frac{ab+dc}{b-c}$ &c.

ey are called *Fractional* or *broken Quantities*.

The *Sign* wherewith *Quantities* are *Connected*, always belongs that *Quantity* which immediately follows it; And therefore the *Quantities concern'd* in any *Queſtion*, may ſtand in any der at *Pleaſure*, viz. the moſt convenient for the *next Operation*. s $a+b-d$ may ſtand thus $b-d+a$. Or thus $a-d+b$ $-d+a+b$ &c. theſe being ſtill the ſame, tho' differently aced.

That *Quantity* which hath no *Sign* before it (as *Generally* e *Leading Quantity* hath not) is always underſtood to have e *Sign* + before it. As a is $+a$ Or $b-d$ is $+b-d$ &c. r the *Sign* + is the *Affirmative Sign*, and therefore all *Leading Poſitive Quantities* are underſtood to have it, as well as thoſe at are to be *Added*.

But the *Sign* — being the *Negative Sign*, or *Sign* of *Defect*, ere is a Neceſſity of *prefixing* it before that *Quantity* to which belongs, where-ever the *Quantity* ſtands.

ct. 2. Of Tracing the Steps uſed in bringing Quantities to an Æquation.

The *Method* of *Tracing* the *Steps*, uſed in bringing the *Quan-ities* concerned in any *Queſtion* to an *Æquation*, is beſt perform'd *Regiſtering* the ſeveral *Operations*, with *Figures* and *Signs* aced in the *Margin* of the Work, according as the ſeveral *Opera-ons* require; being very uſeful in *Long* and *Tedious Operations*.

For *Inſtance*: If it be required to ſet down, and *Regiſter* the *um* of the *Two Quantities* a, and b, the Work will ſtand,

hus	1	a
	2	b
+2	3	$a+b$

First ſet down the propoſed *Quantities*, a and b over-againſt the *Figures* 1. 2. in the *Small Column*, (which are here called *Steps*) and againſt 3 (the *Third Step*) ſet down their *Sum*, viz. $a+b$.

hen againſt that *Third Step*, ſet down 1+2 in the *Margin*; hich Denotes that the *Quantities* againſt the *Firſt* and *Second eps* are added together, and that thoſe in the *Third Step* are eir *Sum*.

To *Illuſtrate* this in *Numbers*, ſuppoſe $a=9$ and $b=6$. hen it will be,

hus	1	$a = 9$
	2	$b = 6$
+2	3	$a+b=9+6=15$ being the *Sum* of 9 and 6.

Again, If it were required to Set down the *Difference* of the same *Two Quantities*, Then it will be,

Thus	1	$a = 9$
	2	$b = 6$
1 — 2	3	$a - b = 9 - 6 = 3$ the Diff. between 9 and 6.

Or if it were required to set down their *Product*. Then it will be,

Thus	1	$a = 9$
	2	$b = 6$
1 × 2	3	$a \times b$ or $ab = 9 \times 6 = 54$ the Prod. of 9 into 6.

&c.

Note, *Letters set or joyned immediately together (like a Word) signify the* Rectangle *or* Product *of those Quantities they Represent. As in the* Last Example, *wherein* $ab = 54$ *is the* Product *of* $a = 9$ and $b = 6$. &c.

Axioms.

1. If *Equal Quantities* be *Added* to *Equal Quantities*, the *Sum* of those *Quantities* will be *equal*.
2. If *Equal Quantities* be *Taken* from *Equal Quantities*, the *Quantities Remaining* will be *equal*.
3. If *Equal Quantities* be *Multiplied* with *Equal Quantities*, their *Products* will be *equal*.
4. If *Equal Quantities* be *Divided* by *Equal Quantities*, their *Quotients* will be *equal*.
5. Those *Quantities*, that are *Equal* to one and the same *Thing*, are *equal* to one *another*.

Note. *I advise the* Learner *to get these five* Axioms *perfectly by Heart*.

These Things being premised, and a perfect Knowledge of the *Signs* and their *Significations* being gained, the Young *Algebraist* may proceed to the following *Rules*. But First I must make bold to *Advise* him here (as I have formerly done) that he be very ready in one *Rule* before he *Undertakes* the *Next*.

That is, He should be *Expert* in *Addition*, before he *meddle* with *Substraction*; And in *Substraction*, before he *undertake* *Multiplication*, &c. because they have a Dependency one upon another.

CHAP.

CHAP. II.

Concerning the Six *Principal* Rules, of Algebraick Arithmetick, of Whole Quantities.

Sect. 1. Addition of Whole Quantities.

Addition of whole *Quantities* admits of *Three Cases.*

Case 1. If the *Quantities* are *Like*, and have *Like Signs*; *Add* the *Co-efficients* or *prefixt Numbers* together; and to their *Sum Adjoyn* the *Quantities* with the same *Sign.*

		Exam. 1.	*Exam.* 2.	*Exam.* 3.	*Exam.* 4.
	1	a	$-a$	$5b$	$-7bc$
	2	a	$-a$	$3b$	$-8bc$
+2	3	$2a$	$-2a$	$8b$	$-15bc$

Thus

		Exam. 5.	*Exam.* 6.	*Exam.* 7.
	1	$3a + 5b$	$3a - 5b$	$6ab+12$
	2	$2a + 7b$	$2a - 7b$	$3ab+24$
+2	3	$5a+12b$	$5a-12b$	$9ab+36$

The Reason of these Additions *is evident from the Work of* Common Arithmetick. *For suppose a, to represent one* Crown, *to which if I Add one* Crown, *the* Sum *will be Two* Crowns, *or* $2a$. *As in Exam.* 1.

Or if we suppose $-a$, *to represent the Want or Debt of One* Crown, *to which if another Want or Debt of One* Crown *be Added, the* Sum *must needs be the Want or Debt of Two* Crowns, *or* $-2a$; *As in Example* 2. *And so for all the rest.*

Case 2. If the *Quantities* are *alike*, And have *unlike Signs*; *Substract* the *Co-efficients* from each other and to *their Difference Adjoyn* the *Quantities* with the *Sign* of the *Greater.*

		Exam. 8.	*Exam.* 9.	*Exam.* 10.	*Exam.* 11.
	1	$+5a$	$-5a$	$7bc$	$-9abd$
	2	$-3a$	$+3a$	$-6bc$	$+7abd$
+2	3	$+2a$	$-2a$	bc	$-2abd$

		Exam. 12.	*Exam.* 13.
	1	$7a-5b$	$-8ab-7bc+15$
	2	$-5a+7b$	$+12ab+7bc-24$
+2	3	$2a+2b$	$4ab-9$

The Reason of the Operations in this Case *may be eas understood by any one that duly considers the comparing of Sto and Debts together, or the Ballancing of Accompts betwi* Debtor *and* Creditor,

That is, The Affirmative Quantities *represent the Stock* Creditor: *The* Negative Quantities *represent the Debts; A their* Sum *represents the Ballance*, &c.

Case 3. When the *Quantities* are *unlike*, Set them all dow without *altering* their *Signs*; and thence will arise *compou Quantities*, which can be no otherwise *Added* but by the *Signs*.

Thus	1	a	a	$5b+7dc$
	2	b	$-b$	$4a-20$
$1+2$	3	$a+b$	$a-b$	$5b+7dc+4a-20$

Here follow a few *Examples* wherein all the 3 *Cases* a promiscuously concerned.

	1	$aa+2ab+bb$	$8ab+bc-37$
	2	$-4ab$	$-7ab-bc+42-6d$
$1+2$	3	$aa-2ab+bb$	$ab+5-6d$

	1	$aa-2ab+bb$	$9bc+7ab-45$
	2	$+4ab+bb$	$4d-6bc-7ab+da$
$1+2$	3	$aa+2ab+bb$	$3bc+4d-45+da$

	1	$5a$	$a+b-ab$
	2	$-7a$	$7c-d$
	3	$+3a$	$4c+f$
$1+2+3$	4	a	$a+b-ab+7c-d+4c+f$

	1	$3aa+4abc-bb+30$	
	2	$2bb-3aa-2abc-25$	
	3	$dd+2aa-3abc-3$	
$1+2+3$	4	$+dd+2aa+bb-abc+2$	

Sect. 2. Substraction of Whole Quantities.

Substraction of Whole *Quantities* is perform'd by one *General Rule.*

Rule.

Change all the Signs of the Substrahend, (viz. *of those Quantities which are to be Substracted*) *or suppose them in your Mind to be changed. Then Add all the Quantities together, as before in Addition, and their Sum will be the true Remainder or Difference required.*

This

This *General Rule* is *deduced* from these evident Truths.

To *Substract* an *Affirmative Quantity*, from an *Affirmative*; the same as to *Add* a *Negative Quantity* to an *Affirmative*.

That is $+2a$ *Taken* from $+3a$, is the same with $-2a$ *dded* to $+3a$.

Consequently, To *Substract* a *Negative Quantity* from an *ffirmative*; will be the same as to *Add* an *Affirmative Quantity* an *Affirmative*.

That is $-2a$ *Taken* from $+3a$ will be the same with $+2a$ *dded* to $+3a$.

		Exam. 1.	Exam. 2.	Exam. 3.	Exam. 4.
	1	$2a$	$-2a$	$8b$	$-15bc$
	2	a	$-a$	$3b$	$-8bc$
-2	3	a	$-a$	$5b$	$-7bc$

		Exam. 5.	Exam. 6.	Exam. 7.
	1	$5a+12b$	$5a-12b$	$9ab+36$
	2	$2a+7b$	$2a-7b$	$3ab+24$
-2	3	$3a+5b$	$3a-5b$	$6ab+12$

		Exam. 8.	Exam. 9.	Exam. 10.	Exam. 11.
	1	$+2a$	$-2a$	bc	$-2abd$
	2	$-3a$	$+3a$	$-6bc$	$+7abd$
-2	3	$+5a$	$-5a$	$+7bc$	$-9abd$

		Exam. 12.	Exam. 13.
	1	$2a+2b$	$4ab-9$
	2	$-5a+7b$	$-8ab-7bc+15$
-2	3	$7a-5b$	$12ab+7bc-24$

If these 13 *Examples* be compared with those in *Addition*; e Work will appear very *evident*, these being only the *Converse* *Proof* of those; according to the *Nature* of *Addition* and *ubstraction* in common *Arithmetick*.

More *Examples* in *Substraction*.

	1	$a+b$	$5bc+3da$	$8a+5bd+25$
	2	$a-b$	$5bc-4da$	$7a-3bd-12$
-2	3	$+2b$	$+7da$	$a+8bd+37$

	1	$c+13$	a	0
	2	$3a-b-2c$	b	$2a-4b$
1 — 2	3	$3c+13-3a+b$	$a-b$	$-2a+4b$

	1	$a+b-54$	
	2	$d-3b-bc-75$	76
			$a-b-5d+7c$
1 — 2	3	$a+4b+bc-21-d$	$76-a+b+5d-7c$

That, $a-b$ Taken from $a+b$ Leaves $+2b$ for the *Remainder*; as in the firſt of theſe *Examples*, may be thus proved:

Let	1	$a+b=z$
And	2	$a-b=x$
2 + b	3	$a=x+b$ *per Axiom* 1.
1 — 3	4	$b=z-x-b$. *per Axiom* 2.
4 + b	5	$2b=z-x$ which was to be proved.

The *Truth* of all *Operations* in *Subſtraction*, where any *Doubt* ariſes, may be *Proved*, by *Adding* the *Subſtrahend* to the *Remainder*; As in *Common Arithmetick*.

Examples.

From	1	$+5a$	0	$-9bc$	
Take	2	$-2a$	$+3b$	$-6da$	*Subſtrahend.*
1 — 2	3	$+7a$	$-3b$	$+6da-9bc$	*Remainder.*
2 + 3	4	$+5a$	0	$-9bc$	*Proof.*

Sect. 3. Multiplication of Whole Quantities.

Multiplication of Whole *Quantities* admits of *Three Caſes*.

Caſe 1. When the *Quantities* have like *Signs*, and no *Coefficients*, ſet or *join* them together, and *prefix* the *Sign* + before them; and that will be their *Product*.

		Exam. 1.	Exam. 2.	Exam. 3.	Exam. 4.
Thus	1	a	$-a$	$a+b$	$-a-b$
	2	b	$-b$	d	$-d$
1 × 2	3	b	$+ab$	$ad+bd$	$+ad+bd$

Caſe 2. If there be *Coefficients*; *Multiply them*, and to their *Product Adjoin* the *Quantities* ſet together as before.

Thus

		Exam. 5.	Exam. 6.	Exam. 7.	Exam. 8.
us	1	$5a$	$-6d$	$3a+2b$	$a+b$
	2	$3b$	$-7b$	6	$5b$
× 2	3	$15ab$	$+42db$	$18a+12b$	$5ab+5bb$

Case 3. When the *Quantities* have *Unlike Signs*; *Join* them d the *Product* of their *Co-efficients* together (as before) but efix the *Sign* — before them;

		Exam. 9.	Exam. 10.	Exam. 11.	Exam. 12.
hus	1	$+a$	$-6d$	$4a+7b$	$4a-7b$
	2	$-b$	$+7b$	$3f$	$-3f$
× 2	3	$-ab$	$-42db$	$12af-21bf$	$-12af+21bf$

hat is, + into +, or — into —, gives + } in the *Product*.
But + into —, or — into +, gives — }

That + into + will *Produce* + in the *Product* is evident om *Multiplication* in *Common Arithmetick*.
z. + 5 into + 7 will give + 35. *&c.*
But that + into —, Or — into + ſhould *Produce* the ign —, As in the *four laſt Examples*,
And that — into — ſhould produce the *Sign* + As in the cond, *fourth*, and *ſixth Examples*, may perhaps ſeem ſomehat hard to be *conceived*; and requires a *Demonſtration*.

Firſt to *prove* that $-7b$ into $+3f = -21bf$. As in *Ex.* 11.

uppoſe	1	$4a-7b=0$
hen will	2	$4a=7b$. *per Axiom* 1.
But	3	$+3f=+3f$
2 × 3	4	$12af=21bf$, *per Axiom* 3.
$-21bf$	5	$12af-21bf=0$. *per Axiom* 2.

Conſequently + into —, Or — into + *Produces* —, which as the Thing to be proved.

Secondly to *prove* $-7b$ into $-3f$ gives $+21bf$ as in *Exam.* 11.

Let	1	$4a-7b=0$ } as before.
Then	2	$4a=7b$ }
But	3	$-3f=-3f$
he 2×3 is	4	$-12af=-21bf$ by what is *proved* above.
$4+21bf$	5	$-12af+bf=0$. *per* Axiom 1.

Conſequently — into — gives + which was to be *proved*.

Or

Or these may be otherwise proved by *Numbers*.

Thus, suppose $\left\{\begin{matrix} a=20 \\ b=14 \end{matrix}\right\}$ and $\left\{\begin{matrix} c=18 \\ d=8 \end{matrix}\right\}$ or any other *Numbers*.

Then $a-b=6$ $c-d=4$ *per* Axiom 2.

Consequently, $a-b \times c-d = 6 \times 4 = 24$ *per* Axiom 3. but $a-b \times c-d$ according to the *Precedent Rules*, will be $ac-cd+bd-da$, which if true must be *Equal* to 24.

Proof $\left\{\begin{matrix} ac=20\times12=240 \\ bd=14\times7=112 \end{matrix}\right.$ $\left\{\begin{matrix} cb=12\times14=168 \\ da=8\times20=160 \end{matrix}\right.$

Hence, $ac+bd=352$ *per* Axiom 1.
And $cb+da=328$ which being *Substracted*,
Leaves $ac+bd-cb-da=352-328=24$ which plainly shews,
That + into − *produces* − }
And − into − *produces* + } in the *Product*.

Q. E. D.

Note, If the *Multiplier* consists of *several Terms*, then every one of those *Terms* must be *Multiplied* into all the *Terms* of the *Multiplied*: And the *Sum* of those *particular Products* will be the *Product* required. As in *Common Arithmetick*.

Examples.

	1	$a+b-d$	$7b+5d$
	2	$a-b$	$3a-5f$
$1 \times a$	3	$aa-ba-da$	$21ba+15da$
$1 \times b$	4	$-ba-bb+db$	$-35bf-25df$
$3+4$	5	$aa-da-bb+db$	$21ba+15da-35bf-25df$

	1	$aa-ba$	$2c-3d$
	2	$a+b$	$3a-4b$
1×2	3	$aaa-abb$	$6ca-9da-8bc+12db$

	1	$aa+2a+4$	$aa-ba+bb$
	2	$a-2$	$a+b$
		$aaa+2aa+4a$	$aaa-baa+bba$
		$-2aa-4a-8$	$+baa-bba+bbb$
1×2	3	$aaa-8$	$aaa+bbb$

Se.

Sect. 4. Diviſion of whole Quantities.

Diviſion of *Species*, is the *converſe* or *direct contrary* to that *Multiplication*, and conſequently performed by *converſe Operations*. (As in *common Arithmetick*) And *admits* of *Four Caſes*.

Caſe 1. When the *Quantities* in the *Dividend*, have *Like Signs* thoſe in the *Diviſor*, and no *Co-efficients* in either; *Caſt off* or *expunge* all the *Quantities* in the *Dividend*, that are like thoſe the *Diviſor*; and ſet down the other *Quantities* with the *Sign* + for the *Quotient* required.

Thus					
	1	ab	$-ab$	$ad+bd$	$-ad-bd$
	2	b	$-b$	d	$-d$
$\div$ 2	3	a	$+a$	$a+b$	$a+b$

Caſe 2. When the *Quantities* in the *Dividend* have *Unlike Signs* to thoſe in the *Diviſor*; then ſet down the *Quotient Quantities* found as before, with the *Sign* — before them.

Thus				
	1	$+ab$	$-ab-bd$	$abc+bcd+bcf$
	2	$-b$	$+b$	$-bc$
$\div$ 2	3	$-a$	$-a-d$	$-a-d-f$

Caſe 3. If the *Quantities* in the *Dividend* and *Diviſor*, have *Co-efficients*; *Divide* the *Numbers* (as in *Common Arithmetick*) and to their *Quotients Adjoin* the *Quotient Quantities*.

Thus				
	1	$15\,ab$	$42\,db$	$12\,af-21\,bf$
	2	$3\,b$	$-7\,b$	$3\,f$
$\div$ 2	3	$5\,a$	$-6\,b$	$4\,a-7\,b$

Note, When the *Quantities* and *Co-efficients* in the *Diviſor* and *Dividend* are all the ſame, the *Quotient* will be an *Unit*, or 1.

Thus					
	1	ab	$9bc$	$7ab+5bc$	$8ab+4d$
	2	ab	$-9bc$	$7ab+5bc$	$-8ab-4d$
$\div$ 2	3	1	-1	1	-1

Caſe 4. When the *Quantities* in the *Diviſor* cannot be exactly found in the *Dividend*; then ſet them both down like a *Vulgar Fraction*. As in *Common Arithmetick*.

Thus {	1	a	$6bc$	$5b+aa$	$8adc$
	2	b	$3d$	$5d+7b$	$4abc$
		a	$2bc$	$5b+aa$	$2d$
$1 \div 2$	3	b	d	$5d+7b$	b

N. B. *In Division one thing must be very carefully observed*, viz. *that Like Signs gives* + *and Unlike Signs gives* — *in the Quotient; which needs no other Proof than that already laid down in the last Section, if duly compared with what hath been said concerning Multiplication and Division, in Vulgar Arithmetick.*

Examples of Division at large.

	1	$21ba+15da-35bf-25df$	$(+3a$
	2	$7b+5d$	
$2\times 3a$	3	$21ba+15da$	
$1-3$	4	$0 \quad 0 \quad -35bf-25df$	$(-5f$
$2\times -5f$	5	$-35bf-25df$	
$4-5$	6	$0 \quad 0$	
$1 \div 2$	7	$3a-5f$ *the Quotient collected from the* 3. *and* 5. *Steps*	

Or *Division* of *Quantities* may stand as *Numbers* in *Common Arithmetick* do; Thus

$$3a-6 \;)\; 6aaaa-96 \;(\; 2aaa+4aa+8a+16$$

$$\begin{array}{r} 6aaaa-12aaa \\ \hline 0+12aaa-96 \\ +12daa-24aa \\ \hline 0+24aa-96 \\ +24aa-48a \\ \hline 0+48a-96 \\ +48a-96 \\ \hline 0 \quad 0 \end{array}$$

That is, $6aaa-96 \div 3a-6$ gives $2aaa+4aa+8a+16$ for the *Quotient*, as may easily be *proved* by *Multiplication*, viz. $2aaa+4aa+8a+16 \times 4a-6$ will *Produce* $6a^4-96$ and so for the rest.

Sect. 5. Involution of whole Quantities.

Involution is the *Raising* or *Producing* of *Powers*, from any proposed *Root*, and is performed in all respects like *Multiplication*, save only in this; *Multiplication admits* of any *different Factors*, but *Involution* still *Retains* the same.

Exam-

Examples.

	1	a	$-a$	*the Root, or ſingle Power.*
⊙ 2	2	aa	$+aa$	*Square, or ſecond Power.*
⊙ 3	3	aaa	$-aaa$	*Cube, or third Power.*
⊙ 4	4	$aaaa$	$+aaaa$	*Biquadrat, or 4th Power.*
⊙ 5	5	$aaaaa$	$-aaaaa$	*Surſolid, or 5th Power, &c.*

Note, *The Figures placed in the Margin, after the Sign* (⊙) *of Involution; ſhew to what height the Root is Involved; and are called Indices of the Power; and are uſually placed over the Involved Quantities, in order to contract the Work, Eſpecially when the Powers are any thing high.*

Thus $\begin{cases} a = a \\ a^2 = aa \\ a^3 = aaa \\ a^4 = aaaa \end{cases}$ And $\begin{cases} a^5 = aaaaa \\ a^6 = aaaaaa \\ a^5\,b^5 = aaaaabbbbb \\ a^3\,b^3\,d^3 = aaabbbddd \end{cases}$

If the *Quantities* have *Co-efficients*, the *Co-efficients* muſt be *Involved* along with the *Quantities*. As in theſe.

Thus	1	$2a$	$-3a$	$5bc$
⊙ 2	2	$4aa$	$+9aa$	$25bbcc$
⊙ 3	3	$8aaa$	$-27aaa$	$125bbbccc$
⊙ 4	4	$16aaaa$	$+81aaaa$	$625b^4c^4$
⊙ 5	5	$32aaaaa$	$-243a^5$	$3125b^5c^5$ &c.

Involution of *Compound Quantities* is performed in the ſame manner, due regard being had to their *Signs* and *Co-efficients*, if there be any.

As for inſtance, Suppoſe $a+b$ were given to be *Involved* to the 5*th Power*.

Thus
1 $a+b$ called a *Binomial Root*.
$a+b$

× a 2 $aa+ab$
× b 3 $+ab+bb$

⊙ 2 4 $aa+2ab+bb$ the *Square* of $a+b$
$a+b$

× a 5 $aaa+2aab+abb$
× b 6 $+aab+2abb+bbb$

⊙ 3 7 $aaa+3aab+3abb+bbb$ the *Cube* of $a+b$

	7	$aaa+3aab+3abb+bbb$
		$a+b$
$7\times a$	8	$a^4+3a^3b+3aabb+abbb$
$7\times b$	9	$+a^3b+3aabb+3abbb+b^4$
1 @ 4	10	$a^4+4a^3b+6aabb+4abbb+b^4$
		$a+b$
$10\times a$	11	$a^5+4a^4b+6a^3bb+4aab^3+ab^4$
$10\times b$	12	$+a^4b+4a^3bb+6aab^3+4ab^4+b^5$
1 @ 5	13	$a^5+5a^4b+10a^3bb+10aab^3+5ab^4+b^5$
		&c.

Again, Let $a-b$, called a *Reſidual Root*, be given.

Then	1	$a-b$
		$a-b$
$1\times a$	2	$aa-ab$
$1\times -b$	3	$-ab+bb$
1 @ 2	4	$aa-2ab+bb$ the *Square* of $a-b$
		$a-b$
$4\times a$	5	$aaa-2aab+abb$
$4\times -b$	6	$-aab+2abb-bbb$
1 @ 3	7	$aaa-3aab+3abb-bbb$ the *Cube* of $a-b$
		$a-b$
$7\times a$	8	$aaaa-3aaab+3aabb-abbb$
$7\times -b$	9	$-aaab+3aabb-3abbb+bbbb$
1 @ 4	10	$aaaa-4aaab+6aabb-4abbb+bbbb$
		$a-b$
$10\times a$	11	$a^5-4a^4b+6a^3bb-4aab^3+ab^4$
$10\times -b$	12	$-a^4b+4a^3bb-6aab^3+4ab^4-b^5$
1 @ 5	13	$a-5a^4b+10a^3bb-10aab^3+5ab^4-b^5$
		&c.

By comparing theſe *Two Examples* together, you may make the following Obſervations.

1. That the *Powers* Raiſed from a *Reſidual Root* (viz. the *Difference* of *Two Quantities*) are the ſame with their *like Powers* Raiſed from a *Binomial Root* (or the *Sum* of *Two Quantities*) ſave only in their *Signs*; viz. the *Binomial Powers* have the *Sign* + to every *Term*; but the *Reſidual Powers* have the *Signs* + and − *interchangeably* to every other *Term*.

2. The *Indices* of the *Powers* of the Leading *Quantity* (*a*) *continually Decreaſe* in *Arithmetical Progreſſion*; viz. in the

e *Square* it is *aa*, *a*, In the *Cube aaa*, *aa*, *a*.
the *Biquadrat* it is *aaaa*, *aaa*, *aa*, *a*, &c.

3. The *Indices* of the other *Quantity b* do *continually increaſe* in *Arithmetical Progreſſion*; viz. In the *Square* it is *b*. *bb*. In the *Cube b*, *bb*, *bbb*. In the *Biquadrat* it is *b*, *bb*, *bbb*, *bbbb*. &c.

4. The *Firſt* and *Laſt Terms*, are always *pure Powers* of the *ingle Quantities*, and are both of the ſame Height.

5. The *Sum* of the *Indices* of any *Two Letters* joined together in the intermediate *Terms*, are always *Equal* to the *Index* of the *ighest Power*, viz. of the *Firſt* or *Laſt Term*.

Theſe *Obſervations* being duly conſidered, it will be eaſy to *onceive* how the *Terms* of any propoſed *Power* raiſed from a *Binomial* or *Reſidual Root* muſt ſtand, without their *Unciæ* or *Numeral Figures*.

For Inſtance, ſuppoſe it were required to *Raiſe* the *Binomial Root* $a+b$ to the *Seventh Power*; then the *Terms* of that *Power* will ſtand without their *Uncia*'s in this Order.

Viz. $a^7+a^6\,b+a^5\,b^2+a^4\,b^3+a^3\,b^4+a^2\,b^5+ab^6+b^7$.

And becauſe the *Unciæ* (not only of any *ſingle Letter*, but alſo) of every *ſingle Power*, how *high* ſoever it be, is an *Unit* or 1 (*which neither Multiplies nor Divides*) and all the *Powers* of any *Binomial*, or *Reſidual Root* are *naturally* raiſed by *Multiplying* of the *Precedent Power* into its *Original Root*, which is done by only *joining* each *Letter* in the *Root* to the *Precedent Power*, with its *Unciæ*, and then removing the ſaid *Power*, when it is ſo *join'd* to the *Second Letter*, one place forwards (either to the *Left*, or *Right Hand*) it muſt needs follow,

That the *Unciæ* of the *Second Terms* (in any ſuch *Power.*) will always be the *Sum* of ſo many *Units Added* together *more one*, as there have been *Multiplications* of the *Firſt Root*; which will always be determined by the *Index* of the *Firſt Term* in the *Power*.

And becauſe the *Uncia*'s of all the *intermediate Terms*, are only removed along with their *Letters*, it alſo follows; that if they are *Added* together, their *reſpective Sums* will *produce* the true *Uncia*'s of the *intermediate Terms* in the new *Raiſed Power*, As doth plainly appear from the following *Numbers* ſo removed without their *Letters*; which both ſhews and *Demonſtrates* an *Eaſy* way of *producing* the *Uncia*'s of any Ordinary *Power* (viz. *of one not very high*) *Raiſed* from either a *Binomial*, or *Reſidual Root*.

Thus

Thus

Add { 1 . 1 . } The two *Uncia*'s of the *first Root.*
{ 1 . 1 }

Add { 1 . 2 . 1 } The *Uncia*'s of the *Square.*
{ 1 . 2 . 1 }

Add { 1 . 3 . 3 . 1 } The *Uncia*'s of the *Cube.*
{ 1 . 3 . 3 . 1 }

Add { 1 . 4 . 6 . 4 . 1 } The *Uncia*'s of the *4th Power.*
{ 1 . 4 . 6 . 4 . 1 }

Add { 1 . 5 . 10 . 10 . 5 . 1 }
{ 1 . 5 . 10 . 10 . 5 . 1 } *Uncia*'s of the *5th Power.*

Add { 1 . 6 . 15 . 20 . 15 . 6 . 1 } *Uncia*'s of the *6th Power.*
{ 1 . 6 . 15 . 20 . 15 . 6 . 1 }

1 . 7 . 21 . 35 . 35 . 21 . 7 . 1 *Unciæ* of the 7 *Pow.*

And so on in this manner *ad infinitum.*

Now if these *Numbers* are *prefix'd* to the aforesaid *Letters*, all the *Terms* will be *compleated* with their respective *Uncia*'s, and will stand thus;

$a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$.

But that the business of finding these *Uncia*'s, may be rendred yet more easy for *Practice*, it will be convenient to consider what *Series* or *Progression*, the *Uncia*'s of each *Term* do make, from the aforesaid *Additions.*

Uncia's of the First Term.	Uncia's of the Second Term.	Uncia's of the Third Term.	Uncia's of the Fourth Term.	Uncia's of the Fifth Term.	Uncia's of the Sixth Term.	Uncia's of the Seventh Term	Uncia's of the Eighth Term, &c.	
.. 1 ..	.. 1 ..	.. ..	.. ..	.. ..	.. ..	.. ..	.. ..	Uncia's of the single Quantity.
1	2	1	.. ..	.. ..	.. ..	.. ..	.. ..	Uncia's of the Square.
1	3	3	1	.. ..	.. ..	.. ..	.. ..	Uncia's of the Cube.
1	4	6	4	1	.. ..	.. ..	.. ..	Uncia's of the 4th Power.
1	5	10	10	5	1	.. ..	.. ..	Uncia's of the 5th Power.
1	6	15	20	15	6	1	.. ..	Uncia's of the 6th Power.
1	7	21	35	35	21	7	1	Uncia's of the 7th Power, &c.

The *Uncia*'s of the *First Term* is only a *Series* of *Units*, whose *Sum* is every where the *Uncia*'s of the *Second Term.*

The *Uncia*'s of the *Second Term*, is a *Series* of *Numbers* in *Arithmetick Progression*; whose Sum is every where the *Uncia*'s of the *Next* Superior *Power* in the *Third Term*, and may be found by *Proposition* 1. Chap. 6. Part 1. That

That is, in the *7th Power* it will be the $\frac{6+1\times 6}{2}=21$ *nciæ* of the *Third Term.*

The reſt of the *Uncia*'s are a Compounded *Series*, whoſe re-pective *Sums* may be obtained from the *Uncia*'s of their *prece-ent Terms.*

Thus $\frac{21\times 5}{3}=35$. Then $\frac{35\times 4}{4}=35$. Again $\frac{35\times 3}{5}=21$

And $\frac{21\times 2}{6}=7$ &c.

From hence may be *deduced* this *General Rule.*

Rule.

If the Index of the Firſt Letter of any Term, be Multiplied nto its own Unciæ, and that Product be Divided by the Number f Terms to that place; the Quotient will be the Unciæ of the ext ſuccecding Term forward.

That is, by the help of thoſe *Indices* that belong to the ſeveral *owers* of the *Firſt* or *Leading Letter* only (as *a*) the true *Uncia*'s f every *Term* may be eaſily underſtood.

Examples.

Let it be required to compleat all the *Terms* of the aforeſaid veral *Powers*, viz. $a^7+a^6b+a^5b^2+a^4b^3+a^3b^4+a^2b^5+ab^6+b^7$ ith their proper *Uncia*'s.

1. The *Index* of a^7 the *Firſt Term* will be the *Uncia* of the *econd Term.* Thus a^7+7a^6b.

2. Then *half* the *Second Terms Index* into its *Uncia*, iz. $\frac{7\times 6}{2}=21$ will be the *Third Terms Uncia.*

Thus $a^7+7a^6b+21a^5b^2$ will be the *Three Firſt Terms.*

3. Again $\frac{21\times 5}{3}=35$ is the *Uncia* of the *Fourth Term.*

Then it will be $a^7+7a^6b+21a^5b^2+35a^4b^3$.

4. And $\frac{35\times 4}{4}=35$ will be the *Uncia* of the *Fifth Term.*

Then $a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4$ &c. until all the *erms* are *compleated* with their reſpective *Uncia*'s; and then ey will ſtand

Thus $a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7$.

Now

Now here it may be *further Obſerv'd*, that the *Uncia*'s do only *increaſe* until the *Indices* of the *Two Letters* become *Equal*, or change places; and then the reſt of the *Uncia*'s will return or *Decreaſe* in the ſame order. That is, wherever the *Indices* of the *Letters* are *Alike*, there the *Uncia*'s will be *Alike*.

And therefore one needs to find the *Uncia*'s (as before) but to *half* the *Number* of *Terms* in any *Power*.

If what hath been ſaid, and the Work of the *Example* be well underſtood, I *preſume* it will be found very eaſy to Raiſe any *Power* from a *Binomial* or *Reſidual Root*, to what *height* you pleaſe; without the trouble of a *continued Involution*; and without the help of ſuch a *Table* of *Powers* as is propoſed by Mr. *Oughtred* in his *Key* to the *Mathematicks*, Page 40. and ſince by others.

Now from theſe *Conſiderations* it was, that I propoſed this *Method* of Raiſing *Powers* in my *Compendium* of *Algebra*, Page 57, as wholly *New* (viz. ſo much of it as was there uſeful) having then (I profeſs) neither ſeen the way of doing it, nor ſo much as heard of its being done. But ſince the writing of that *Tract*, I find in Dr. *Wallis*'s *Hiſtory of Algebra*, *Page* 319 and 331, that the Learned Mr. *Iſaac Newton* had diſcovered it long before: which the *Doctor* ſets down in this *manner*.

Let *m* be the *Exponent* of the *Power*.

Then $\left\{ \begin{array}{l} m-0 \quad m-1 \quad m-2 \quad m-3 \quad m-4 \\ 1 \times \overset{1}{\text{——}} \times \overset{2}{\text{——}} \times \overset{3}{\text{——}} \times \overset{4}{\text{——}} \times \overset{5}{\text{——}} \&c. \end{array} \right.$

Will be the *Series* of the *Uncia*'s required; but he doth not tell us how they firſt came to be found out, nor have I ever met with the leaſt hint of it in any Author.

Sect. 6. Evolution of whole Quantities.

Evolution is the *Extracting* of *Roots* from any given *Power*. That is, it is the Converſe Work to that of *Involution*, and in *ſingle Quantities* it's eaſy, if the given *Power* have ſuch a *Root* as is required, which may be thus known.

If the given *Power* have no *Numbers prefix'd* to it, and its *Index* can be *Divided* by the *Index* of the *Root* required, the *Quotient* will be the *Index* of the *Root* ſought.

Thus, If the *Cube Root* of *aaaaaa*, viz. a^6 were required (the *Index* of the *Cube* is 3) then 3) 6 (2. That is, 3) a^6 (a^2 the *Root* required. And ſuch *Operations* are uſually ſet down Thus

hus	1	a^6	a^6b^6	$a^6b^6d^6$
w 2	2	a^3	a^3b^3	$a^3b^3d^3$
w 3	3	aa	$aabb$	$aabbdd$
w 2	4	a	ab	abd

Note, *The Figures placed in the Margin after the Sign* (ww) *of olution, denote the Index of the Root to be Extracted.*

If the given *Powers* have *Co-efficients* : (viz. *Numbers pre-d* to them ;) then you muſt *Extract* their reſpective *Roots* in *Vulgar Arithmetick.*

Thus	1	$81a^4$	$1296a^8b^8$	$20736a^4b^4c^4$
1 ww 2	2	$9aa$	$36a^4b^4$	$144aabbcc$
1 ww 4	3	$3a$	$6aabb$	$12abc$
2ww2	4	$3a$	$6aabb$	$12abc$

But if the *Root* required cannot be truly *Extracted* out of both e *Co-efficients* and *Indices* of the given *Power* ; then it is a *rd*, and muſt have the *Sign* of the *Root* required *prefix'd* to it.

Thus	1	a^5	$67aaaa$	$216bbbddd$
ww 2	2	$\sqrt{a^5}$	$\sqrt{67aaaa}$	$\sqrt{216bbbddd}$
ww 3	3	$\sqrt[3]{a^5}$	$\sqrt[3]{67aaaa}$	$6bd$

Evolution of *Compound Quantities* or *Powers*, is a little more oubleſome than that of *Single Powers* ; and would require a eat many Words to *Explain* the manner, and *Reaſon* of form-g the ſeveral *Canons*, that are commonly uſed in *Extracting* e *Roots* of *Compound Quantities* ; eſpecially if the *Powers* be ry *high*, &c. I ſhall therefore for brevity's ſake omit them, d *inſtead* thereof propoſe an *eaſy Method* of diſcovering the oots of all *Compound Powers* in general ; And in order to that, will be neceſſary to premiſe ; that if either the *Sum* or *Diffe-nce* of ſeveral *Quantities* be *Involved* to any *Power*, there ill ariſe ſo many *ſingle Powers* of the ſame *height*, as there are *Different Quantities.*

As for inſtance, if $a+b+d$ be *Squared* ; that is, be *Involved* the *Second Power*, it will be $aa+2ab+2ad+bb+2bd+dd$, ere you have *aa*, *bb*, and *dd*.

Again, if $a+b+d$ were *cubed*, viz. *Involved* to the *Third Power*, then you will have *aaa*, *bbb*, and *ddd*, in it, &c.

Whence it follows that in *Extracting* the *Roots* of all *Compound Quantities*, there must be consider'd,

1. How many *Different Letters* (or *Quantities*) there are in the given *Power*.

2. Whether the *Single Powers* of each of those *Letters* be of an *Equal height*, and have in them such a *single Root* as is required: Which if they have, *Extract* it as before.

3. Connect those *single Roots* together with the *Sign* +, and *Involve* them to the same *height* with the given *Power*; this being done, compare the new *Raised Power* with the given Power, and if they are alike in all their respective *Terms*, then you have the *Root* required; or if they *Differ* only in their *Signs* the *Root* may be *Easily corrected* with the *Sign* — as occasion requires.

Example 1. Let it be required to *Extract* the *Square Root* of $cc+2cb-2cd+bb-2bd+dd$.

In this *Compound Square* there are *Three distinct Powers* viz. bb, cc, dd, whose *single Roots* are b, c, d, wherefore I suppose the *Root sought* to be $b+c+d$, or rather $b+c-d$, because in the given *Power* there is $-2cd$, and $-2bd$, therefore I conclude it is $-d$, then $b+c-d$ being *Squared*, *produces* $bb+\ldots-2bd+cc-2cd+dd$ which I find to be the same in all its *Terms* with the given *Power*, although they stand in a *Different Position*; consequently $b+c-d$ is the true *Root* required.

Example 2. 'Tis required to *Extract* the *Square Root* of $a^4-2aabb+b^4$. Here are but Two *single Powers*, viz. a^4 and b^4, whose *Square Roots* are aa, and bb. And because in the given *Power* there is $-2aabb$, therefore I conclude it must either be $aa-bb$, or $bb-aa$. Both which being *Involved*, will *produce* $a^4-2aabb+b^4$ consequently the *Root sought* may either be $aa-bb$, or $bb-aa$ according to the *Nature* or *design* of the *Question*, from whence the given *Power* was *produced*.

Example 3. Let it be required to *Extract* the *Square Root* of $36aaaa+108aa+81$. Here the *Two single Powers* are $36aaaa$ and 81, whose *Roots* are $6aa$ and 9. And because the *Signs* are all + therefore I suppose the *Root* to be $6aa+9$, the which being *Involved* doth *produce* $36a^4+108aa+81$, consequently $6aa+9$ is the true *Root* required.

Example 4. Suppose it were required to *Extract* the *Square Root* of $125aaa+300aae-450aa+250aee-720ae+640e+540a+288ee+432e-216$.

n this *Example* there is *Three distinct Powers*, viz. $125aaa$, $64eee$, and -216.

The *Cube Root* of $125aaa$ is $5a$. Of $64eee$ is $4e$, and the *Cube* ot of -216 is -6.

Wherefore I suppose the *Root sought* to be $5a+4e-6$, which ng *Involved* to the *Third Power*, does *produce* the same with given *Power*, consequently $5a+4e-6$ is the *Cube Root* uired.

ut if the new *Power*, raised from the *supposed Root (being olved to its due height)* should not *prove* the same with the n *Power*, viz. if it hath either *More*, or *Fewer Terms* in it,

Then you may *conclude* the given *Power* to be a *Surd*, and t have its proper *Sign prefix'd* to it, and cannot be otherwise pres'd, until it come to be *Evolved* in *Numbers*.

Example 5. Suppose it were required to *extract* the *Cube* ot of $27aaa+54baa+8bbb$.

Here are *Two distinct* and *perfect Cubes*, viz. $27aaa$, and $8bbb$, ose *Cube Roots* are $3a$ and $2b$.

Wherefore one may suppose the *Root sought* to be $3a+2b$ ch being *Involved* to the *Third Power*, is $27aaa+54baa+36bba+8bbb$. Now this *new raised Power* hath one *Term* z. $36bba$) more in it than the given *Power* hath; but this ng a *perfect Cube*, one may therefore *conclude* the given *Power* ot so, *viz.* it is a *Surd*, and hath not such a *Root* as was *Required, but must be Expressed*, or set down,

Thus $\sqrt[3]{27aaa+54baa+8bbb}$.

f these *Examples* be well understood, the *Learner* will find very *easy* by this *Method* of *proceeding* to *discover* the true ot of any given *Power* whatsoever.

CHAP. III.

Of Algebraick Fractions, or Broken Quantities.

Sect. 1. Notation of *Fractional Quantities*.

Fractional Quantities are *Express'd* or set down like *Vulgar ractions* in *common Arithmetick*.

Thus $\left\{\frac{a}{b}, \frac{2bc}{d}, \frac{5b-4a}{4d+7b}\right.$ *Numerators.* / *Denominators.*

How they come to be ſo, ſee *Caſe* 4. in the laſt *Chapter* of *Diviſion*. Theſe *Fractional Quantities* are managed in all *reſpects* like *Vulgar Fractions* in *Common Arithmetick*.

Sect. 2. To **Alter**, or **Change** different **Fractions** into one *Denomination*, retaining the ſame Value.

Rule.

Multiply all the Denominators into each other for a New Denominator; and each Numerator into all the Denominators but its own, for New Numerators.

Examples.

Let it be required to bring $\frac{a}{b}$ and $\frac{d}{c}$ into one *Denomination*.

Firſt $a \times c$, and $d \times b$, will be the *Numerators*, and $b \times c$ will be the *common Denominator*, viz. $\frac{ca}{bc}$ and $\frac{bd}{bc}$ are the *Two Fractions* required. That is $\frac{ca}{bc} = \frac{a}{b}$ and $\frac{bd}{bc} = \frac{d}{c}$.

Again, let $\frac{b+c}{a+b}$ and $\frac{d-c}{b-d}$ be brought in one *Denomination*.

And they will be $\left\{\frac{bb+bc-bd-dc}{ba+bb-da-bd}\right.$ and $\frac{ad-ac+bd-bc}{ba+bb-da-bd}$ &c.

Sect. 3. To **Bring** whole **Quantities** into **Fractions** of a given *Denomination*.

Rule.

Multiply the whole Quantities into the given Denominator for a Numerator; under which Subſcribe the given Denominator, and you will have the Fraction required.

Examples.

Let it be required to bring $a+b$ into a *Fraction*, whoſe *Denominator* is $d-a$, *Firſt* $a+b \times d-a$ is $da+bd-aa-ba$.

Then $\frac{da+bd-aa-ba}{d-a}$ is the *Fraction* required.

Again $b+\frac{a}{d}$ will be $\frac{db+a}{d}$. And $\frac{aa}{d}-a$ will be $\frac{aa-da}{a}$.

Alſo $a+b+\frac{aa+bb}{a-b}$ will be $\frac{2aa}{a-b}$.

Whe

When whole *Quantities* are to be ſet down *Fraction-wiſe*, ubſcribe an *Unit* for the *Denominator*.

Thus ab, is $\frac{ab}{1}$ And $aa-bb$, is $\frac{aa-bb}{1}$ &c.

Sect. 4. To **Abbreviate**, or **Reduce** *Fractional Quantities* into their Loweſt *Denomination*.

Rule.

Divide both the Numerator and Denominator by their greateſt ommon Diviſor, viz. *by ſuch Quantities as are found in both*; *nd their Quotients will be the Fraction in its Loweſt Term.*

Thus $\frac{aac}{dc}$ is $\frac{aa}{d}$. And $\frac{abbb}{abc}$ is $\frac{bb}{c}$. Alſo $a+\frac{bdc}{bc}=a+d$.

In ſuch *ſingle Fractions* as theſe; the *common Diviſors* (if there e any) are *eaſily* diſcover'd by *Inſpection* only; but in *Compound ractions* it often *proves* very *troubleſome*, and muſt be done ither by *Dividing* the *Numerator* by the *Denominator*, until nothing *Remains*, when that can be done: Or elſe finding *their ommon Meaſure*; by *Dividing* the *Denominator* by the *Nume-rator*, and the *Numerator* by the *Remainder*, and ſo on as in *Vulgar Fractions* (*Sect.* 4. *Page* 51.)

Examples.

Suppoſe $\frac{aac-aad}{cd-dd}$ were to be *reduced Lower*.

Then $cd-dd)\ aac-aad\ (\frac{aa}{d}$ the *Fraction required*.

$aac-aad$

o o

In this *Example* it ſo *happens* that the *Numerator* is *Divided juſt Off* by the *Denominator*; but in the *Next* it's otherwiſe, and requires a *Double Diviſion* to find out the *common Meaſure*. *Viz.*

Let it be required to *Reduce* $\frac{aaa-abb}{aa+2ab+bb}$ to its *Loweſt Terms*.

Firſt $aa+2ab+bb)\ aa-abb\ (a$

$aaa+2aab-abb$

$-2aab-2abb$ the *Remainder*.

Then $-2aab-2abb)\ aa+2ab+bb\ (-\frac{1}{2b}-\frac{1}{2a}$ &c.

$aa+ab$

$ab+bb$

$ab+bb$

o o

Hence

Hence it appears that $-2aab-2abb$ is the *common Measure*; by which $aaa-abb$ being *Divided*.

Viz. $-2aab-2abb)\ aaa-abb\ (-\frac{a}{2b}+\frac{1}{2}$

$aaa+aab$

$-aab-abb$

$-aab-abb$

$0\quad 0$

Then $-\frac{a}{2b}+\frac{1}{2}$ is the *New Numerator*.

And ☞ $-\frac{1}{2b}+\frac{1}{2a}$ is the *New Denominator*.

But $-\frac{1}{2b}+\frac{1}{2}=-\frac{2a+2b}{4b}=-\frac{a+b}{2b}$ the *Numerator*.

And $-\frac{1}{2b}-\frac{1}{2a}=-\frac{2a+2b}{4ba}=-\frac{a-b}{2ba}$ the *Denominator*.

Let both be *Multiplied* with $2ba$ and you will have

$-aa+ab$ the *Numerator*.

$-a-b$ the *Denominator*. Or *Changing* the *Signs* of all the *Quanties*, and it will be $\frac{aa-ab}{a+b}$ the *New Fraction* Required. That is $\frac{aa-ab}{a+b}=\frac{aaa-abb}{aa+2ab+bb}$

Again, let it be required to *Reduce* $\frac{dd-bb}{ddd-bbb}$

The *Common Measure* of this *Fraction* will be the *easiest* found (as *appears* from *Trials*) by *dividing* the *Denominator* by the *Numerator*, &c.

Thus $dd-bb)\ ddd-bbb\ (d$

$ddd-bbd$

$+bbd-bbb)\ dd-bb\ (\frac{d}{bb}$

$dd-bd$

$+bd-bb)\ bbd-b^3\ (b$

$bbd-b^3$

$0\quad 0$

Hence it *appears* that $bd-bb$ is the *common Measure* that will *Divide* both the *Numerator* and the *Denominator*.

Conse-

Conſequently $bd-bb)\ dd-bb\ (\frac{d}{b}+1$. is the *New Numerator.*

$$\begin{array}{r} dd-db \\ \hline +db-bb \\ db-bb \\ \hline 0\quad 0 \end{array}$$

And $bd-bb)\ ddd-bbb\ (\frac{dd}{b}+d+b$ the *New Denominator.*

$$\begin{array}{r} ddd-ddb \\ \hline +ddb-bbb \\ ddb-bbd \\ \hline +bbd-bbb \\ bbd-bbb \\ \hline 0\quad 0 \end{array}$$

Let both be *Multiplied* with b, and then you will have $\left\{\begin{array}{ll} \frac{d+b}{dd+bd+bb} & \text{the } \textit{Numerator}, \\ & \text{the } \textit{Denominator}, \end{array}\right\}$ of the *Fraction* required.

But if after all *Means* uſed *(as above)* there cannot be found *ne Common Meaſure* to both the *Numerator* and *Denominator*, hen is that *Fraction* in its *Leaſt Terms* already.

Note, *Theſe Operations will be underſtood by a Learner after e hath paſs'd thro' Multiplication, and Diviſion of Fractions.*

Sect. 5. Addition and Subſtraction of *Fractional Quantities.*

The given *Fractions* being of *One Denomination*, or if they re not, make them ſo, *per Sect.* 4. Then,

Rule.

Add, or Subſtract their Numerators, as Occaſion requires, and o their Sum or Difference, Subſcribe the common Denominator: As in Vulgar Fractions.

Examples in Addition.

	1	$\frac{bb}{c}$	$\frac{a+b}{d}$	$\frac{2a-b}{d+c}$	$\frac{a-b+d}{d+a}$
	2	$\frac{aa}{c}$	$\frac{2a+c}{d}$	$\frac{2b-a}{d+c}$	$\frac{a+b-d}{d+c}$
1 + 2	3	$\frac{bb+aa}{c}$	$\frac{3a+b+c}{d}$	$\frac{a+b}{d+c}$	$\frac{2a}{d+a}$

Exam-

Examples in Subtraction.

	1	$\frac{bb+aa}{c}$	$\frac{a+b}{d+c}$	$\frac{3a+b+c}{d}$	$\frac{2b}{d+a}$
	2	$\frac{bb}{c}$	$\frac{2b-a}{d+c}$	$\frac{2a+c}{d}$	$\frac{a+b-d}{d+a}$
1—2	3	$\frac{aa}{c}$	$\frac{2a-b}{d+c}$	$\frac{a+b}{d}$	$\frac{b-a+d}{d+a}$

Sect. 6. Multiplication of *Fractional Quantities.*

First *prepare Mix'd Quantities (if there be any)* by making them *Improper Fractions*, and *Whole Quantities* by *subscribing* an *Unit* under them. As *per Sect.* 3. Then,

Rule.

Multiply the Numerators together for a new Numerator: And the Denominators together for a new Denominator. As in Vulgar Fractions.

Thus	1	$\frac{ab}{c}$	$\frac{3a-2b}{2d+c}$
	2	$\frac{d}{f}$	$\frac{4a+2b}{d}$
1 × 2	3	$\frac{abd}{cf}$	$\frac{12aa-2ab-4bb}{2dd+dc}$

Suppose it were required to *Multiply* $2a+\frac{b}{c}-25$ with $3b+4c$. These *prepared* for the Work *(per Sect.* 3.) will stand

Thus {	1	$\frac{2ac+b-25c}{c}$
	2	$\frac{3b+4c}{1}$
1 × 2	3	$\frac{6bac+3bb-75bc+8acc+4bc-100cc}{c}$
Or	4	$6ba-71b+8ac-100c+\frac{3bb}{c}$ *per Sect.* 4.

N. B. Any *Fraction* is *Multiplied* with its *Denominator* by *ſting Off*, or *Taking* the *Denominator* away.

Thus $\frac{b}{a} \times a$ gives b. For $\frac{b}{a} \times \frac{a}{1} = \frac{ba}{a} = b$ &c.

Sect. 7. Diviſion of *Fractional Quantities.*

The *Fractional Quantities* being *prepared*, as Directed in the *Section.* Then,

Rule.

Multiply the Numerator of the Dividend, into the Denomi-tor of the Diviſor, for a New Numerator; and Multiply the her two together for a New Denominator; as in Vulgar Fract.

Examples.

Let $\frac{abd}{cf}$ be *Divided* by $\frac{ab}{c}$ the Work may ſtand

Thus $\frac{ab}{c}) \quad \frac{abd}{cf} \quad (\frac{abdc}{abcf} = \frac{d}{f}$ *per Sect.* 4.

r Thus	1	$\frac{abd}{cf}$	$\frac{a+b}{d}$	$\frac{aaa-bbb}{a+b}$
		$\frac{ab}{c}$	$\frac{c-b}{a}$	$\frac{aa-ab+bb}{c}$
	2	$\frac{c}{d}$	$\frac{a}{aa+ba}$	$\frac{c}{aaac-bbbc}$
1 ÷ 2	3	$\frac{d}{f}$	$\frac{aa+ba}{dc-db}$	$\frac{aaac-bbbc}{aaa+bbb}$

Suppoſe it were required to *Divide* $aa + \frac{3abb}{a+4b}$

By $a+b$. Then the Work *prepared* will ſtand

Thus $\frac{a+b}{1}) \quad \frac{aaa+4aab+3abb}{a+4b} \quad (\frac{aaa+4aab+3abb}{aa+5ba+4bb}$

But $\frac{aaa+4aab+3abb}{aa+5ba+4bb} = \frac{aa+3b}{a+4b}$ *(per Sect.* 4.)

When *Fractions* are of *One Denomination*, *caſt Off* the *De-ominators*, and *Divide* the *Numerators.*

Thus, if $\frac{ab^3}{c}$ were to be *Divided* by $\frac{bb}{c}$ it will be

$bb) ab^3 (ab$ the *Quotient* required.

For $\frac{bb}{c}$) $\frac{ab^3}{c}$ ($\frac{ab^3c}{bbc}$ But $\frac{ab^3c}{bbc} = ab$ (*per Sect.* 4.)

Again, Suppose it were required to *Divide* $\frac{a^3-abb}{c-d}$

By $\frac{aa+2ab+bb}{c-d}$ *Casting Off* $c-d$ into both, it will be

$aa+2ab+bb)\ aaa-abb$ ($\frac{aa-ba}{a+b}$ &c.

Sect. 8. Involution of *Fractional Quantities.*

Rule.

Involve the Numerator into it self, for a New Numerator, and the Denominator into it self, for a New Denominator; and as often as the Power requires.

Thus	1	$\frac{b}{a}$	$\frac{3bc}{2ad}$	$\frac{b+d}{a-c}$
1 @ 2	2	$\frac{bb}{aa}$	$\frac{9bbcc}{4aadd}$	$\frac{bb+2bd+dd}{aa-2ac+cc}$
1 @ 3	3	$\frac{bbb}{aaa}$	$\frac{27bbbccc}{8aaaddd}$	$\frac{bbb+3bbd+3bdd+ddd}{aaa-3aac+3acc-ccc}$

Sect. 9. Evolution of *Fractional Quantities.*

If the *Numerator* and *Denominator* of the given *Fraction* have each of them such a *Root* as is required; *(which very rarely happens)* Then *Evolve* them; and their *respective Roots* will be the *Numerator* and *Denominator* of the new *Fraction* required.

Thus	1	$\frac{9aabb}{4dd}$	$\frac{aa+2ab+bb}{aa-2ab+bb}$
1 ω 2	2	$\frac{3ab}{2d}$	$\frac{a+b}{a-b}$
Again	1	$\frac{27aaabbb}{8ddd}$	$\frac{aaa+3aab+3abb+bbb}{aaa-3aab+3abb-bbb}$
1 ω 3	2	$\frac{3ab}{2d}$	$\frac{a+b}{a-b}$

Sometimes it so falls out, that the *Numerator* may have such a *Root* as is required, when the *Denominator* hath not

the *Denominator* may have ſuch a *Root*, when the *Numera-* hath not. In thoſe *Caſes* the *Operations* may be ſet down

hus	1	$\frac{aabb}{ddd}$	$\frac{aaa+4bb-dd}{aa+2ab+bb}$
		$\frac{ab}{\sqrt{ddd}}$	$\frac{\sqrt{aaa+4bb-dd}}{a+b}$
ow	2		

But when neither the *Numerator*, nor the *Denominator* have ſt ſuch a *Root* as is required; *prefix* the *Radical Sign* of the oot to the *Fraction*; and then it becomes a *Surd*, as in the laſt ep, which brings me to the Buſineſs of *managing Surds*.

CHAP. IV.

Of Surd Quantities.

THE whole *Doctrine* of *Surds* (*as they call it*) were it fully handled, would require a very *large Explanation* (to render but *Tolerably Intelligible*) even enough to fill a *Treatiſe* it lf; if all the *Various Explanations* that may be of uſe to make *eaſy* ſhould be *inſerted*; without which it's very *intricate* and oubleſome for a *Learner* to underſtand. But now theſe *tedious eductions* of *Surds*, which were heretofore thought *uſeful* to *fit Equations* for ſuch a *Solution*, as was then *underſtood*, are wholly id aſide as *uſeleſs*: Since the *New Methods* of *Reſolving* all ſorts *Æquations* renders their *Solution equally eaſy*, although their *owers* are never ſo *high*.

Nay, even ſince the true Uſe of *Decimal Arithmetick* hath been ell *underſtood*, the Buſineſs of *Surd Numbers* has been managed at way; as *appears* by ſeveral *Inſtances* of that kind in *Doctor Vallis's History of Algebra*, from *Page* 23 to 29.

I ſhall therefore, for *Brevity ſake*, paſs over thoſe tedious *Reductions*, and only ſhew the *Young Algebraiſt* how to deal ith ſuch *Surd Quantities* as may ariſe in the *Solution* of *hard Queſtions*.

Sect. 1. Addition and Subſtraction of *Surd Quantities*.

Caſe 1. When the *Surd Quantities* are *Homogeneal* (*viz.* *re alike*) *Add*, or *Subſtract* the *Rational Part*, if they are

joined to any, and to their *Sum*, or *Difference*, *Adjoin* th *Irrational* or *Surd*.

Examples in **Addition.**

	1	$5\sqrt{bc}$	$6b\sqrt{ac}$	$b\sqrt{aa+cc}$
	2	$7\sqrt{bc}$	$4b\sqrt{ac}$	$3b\sqrt{aa+cc}$
1 + 2	3	$12\sqrt{bc}$	$10b\sqrt{ac}$	$4b\sqrt{aa+cc}$

	1	$4d^3\sqrt{aa}$	$b+\sqrt[3]{aa-cc}$	$bc^5\sqrt{aa+d}$
	2	$d^3\sqrt{aa}$	$c-\sqrt[3]{aa-cc}$	$3bc^5\sqrt{aa+d}$
1 + 2	3	$5d^3\sqrt{aa}$	$b+c$	$4bc^5\sqrt{aa+d}$

Examples in **Subſtraction.**

	1	$12\sqrt{bc}$	$10b\sqrt{ac}$	$4b\sqrt{aa+cc}$
	2	$7\sqrt{bc}$	$4b\sqrt{ac}$	$3b\sqrt{aa+cc}$
1 − 2	3	$5\sqrt{bc}$	$6b\sqrt{ac}$	$b\sqrt{aa+cc}$

	1	$5d^3\sqrt{aa}$	$b+c$	$4bc^5\sqrt{aa+d}$
	2	$4d^3\sqrt{aa}$	$c-\sqrt[13]{aa-cc}$	$3bc^5\sqrt{aa+d}$
1 − 2	3	$d^3\sqrt{aa}$	$b+\sqrt[3]{aa-cc}$	$bc^5\sqrt{aa+d}$

Caſe 2. When the *Surd Quantities* are *Heterogeneal* (viz. *their Indices are unlike*) they are only to be *Added*, or *Subſtracted* by their *Signs*, viz. + Or − And from thence will ariſe *Surd* either *Binomial*, or *Reſidual*.

Examples in **Addition.**

	1	$\sqrt{bc}$	$4d\sqrt{a}$	$\sqrt[3]{ac-ba}$
	2	$\sqrt{ba}$	$3b\sqrt{ac}$	$\sqrt{ac+ba}$
1 + 2	3	$\sqrt{bc}+\sqrt{ba}$	$4d\sqrt{a}+2b\sqrt{ac}$	$\sqrt[3]{ac-ba}+\sqrt{ac+ba}$

Examples in **Subſtraction.**

	1	$\sqrt{bc}$	$b-d\sqrt{aaa+ca}$
	2	$\sqrt{ba}$	$d-2a\sqrt{bd+dd}$
1 − 2	3	$\sqrt{bc}-\sqrt{ba}$	$b-d\sqrt{aaa+ca}-d+2a\sqrt{bd+dd}$

Se

Sect. 2. Multiplication of *Surd Quantities.*

Case 1. When the *Quantities* are *Pure Surds* of the ſame *kind*; *Multiply* them together, and to their *Product prefix* their *Radical Sign.*

Examples.

	1	$\sqrt{b}$	$\sqrt{ba+da}$	$\sqrt{aa+bb}$
	2	$\sqrt{a}$	$\sqrt{ca}$	$\sqrt{aa-bb}$
× 2	3	$\sqrt{ba}$	$\sqrt{bcaa+dcaa}$	$\sqrt{aaaa-bbbb}$

Case 2. If *Surd Quantities* of the ſame *kind (as before)* are *joined* to *Rational Quantities*, then *Multiply* the *Rational* into the *Rational*; and the *Surd* into the *Surd*, and *join* their *Products* together.

Examples.

	1	$d\sqrt{bc}$	$5cd\sqrt{ba+da}$	$15\sqrt{ab}$
	2	$3b\sqrt{a}$	$3a\sqrt{ca}$	$5\sqrt{d}$
× 2	3	$3db\sqrt{bca}$	$15cda\sqrt{bcaa+dcaa}$	$75\sqrt{abd}$

Sect. 6. Diviſion of *Surd Quantities.*

Case 1. When the *Quantities* are *Pure Surds* of the ſame *kind*, and can be *Divided Off* (viz. *without leaving a Remainder*) *Divide* them, and to their *Quotient prefix* their *Radical Sign.*

Examples.

	1	$\sqrt{ba}$	$\sqrt{bcaa+dcaa}$	$\sqrt{aaaa-bbbb}$
	2	$\sqrt{b}$	$\sqrt{ca}$	$\sqrt{aa-bb}$
÷ 2	3	$\sqrt{a}$	$\sqrt{ba+da}$	$\sqrt{aa+bb}$

Case 2. If *Surd Quantities*, of the ſame *kind*, are *joined* to *Rational Quantities*; then *Divide* the *Rational* by the *Rational*, if it can be, and to their *Quotient*, *join* the *Quotient* of the *Surd*, *Divided* by the *Surd*, with its Firſt *Radical Sign.*

Examples.

	1	$3db\sqrt{bca}$	$15cda\sqrt{bcaa+dcaa}$	$75\sqrt{abd}$
	2	$3b\sqrt{a}$	$3a\sqrt{ca}$	$5\sqrt{d}$
÷ 2	3	$d\sqrt{bc}$	$5cd\sqrt{ba+da}$	$15\sqrt{ab}$

Note,

Note, If any *Square* be *Divided* by its *Root*, the *Quotient* will be its *Root*.

Examples.

	1	a	$bb+2bc+cc$	$aaaa-2bbaa+bbbb$
	2	$\sqrt{a}$	$\sqrt{bb+2bc+cc}$	$\sqrt{a^4-2bbaa+b^4}$
$1 \div 2$	3	$\sqrt{a}$	$\sqrt{bb+2bc+cc}$	$\sqrt{a^4-2bbaa+b^4}$

Sect. 4. Involution of *Surd Quantities*.

Case 1. When the *Surds* are not *joined* to *Rational Quantities* they are *Involved* to the ſame height as their *Index denotes*, by only taking away their *Radical Sign*.

Examples.

	1	$\sqrt{a}$	$\sqrt{bca}$	$\sqrt{aa-bb}$	$\sqrt{5a-da}$
$1 \odot 2$	2	a	bca	$aa-bb$	$5a-da$

Caſe. 2. When the *Surds* are *joined* to *Rational Quantities*, *Involve* the *Rational Quantities* to the ſame *height* as the *Index* of the *Surd denotes*; then *Multiply* thoſe *Involved Quantities* into the *Surd Quantities*, after their *Radical Sign* is taken away. As before.

Examples.

	1	$b\sqrt{a}$	$5d\sqrt{ca}$	$3b\sqrt{aa-dd}$
$1 \odot 2$	2	bba	$25ddca$	$9bbaa-9bbdd$

	1	$a^3\sqrt{bc}$	$3d\sqrt{aa+bb}$	$da^3\sqrt{b}$
$1 \odot 3$	2	$aaabc$	$27dddaa+27dddbb$	$dddaaab$

The *Reaſon* of only taking away the *Radical Sign*, as in *Caſe* 1. is eaſily conceived, if you conſider that any *Root*, being *Involved* into it ſelf, *produces* a *Square*, &c.

And from thence the *Reaſon* of thoſe *Operations* performed by the *Second Caſe* may be thus Stated.

Suppoſe $b\sqrt{a}=x$. Then $\sqrt{a}=\frac{x}{b}$ *per Axiom* 4. and both *Sides* of the *Æquation* being *Equally Involved*, it will be $a=\frac{xx}{bb}$.

Then *Multiplying* both *Sides* of the *Æquation* into bb, it becomes $bba=xx$ *per Axiom* 3.

Which was to be proved.

Again

Again, Let $5d\sqrt{ca}=x$. Then $\sqrt{ca}=\frac{x}{5d}$

And $ca=\frac{xx}{25dd}$ Conſequently $25ddca=xx$

Alſo from hence it will be eaſy to *deduce* the *Reaſon* of *Multiplying Surd Quantities*, according to both the *Caſes*. For

Suppoſe {	1	$\sqrt{b}=z$ } *Example* 1. *Caſe* 1.
	2	$\sqrt{a}=x$ }
1 @ 2	3	$b=zz$
2 @ 2	4	$a=xx$
3 × 4	5	$ba=zzxx$. *per Axiom* 3.
5 ww 2	6	$\sqrt{ba}=zx$. which was to be *proved*.

et {	1	$d\sqrt{bc}=z$ } *Example* 1. *Caſe* 2.
	2	$3b\sqrt{a}=x$ }
$1\div d$	3	$\sqrt{bc}=\frac{z}{d}$
$\div 3b$	4	$\sqrt{a}=\frac{x}{3b}$
4 × 3	5	$\sqrt{bca}=\frac{zz}{3bd}$ from what is *proved* above.
$\times 3bd$	6	$3bd\sqrt{bca}=zx$, &c. for the reſt.

Diviſion being the *Converſe* to *Multiplication*, needs no other *Proof*.

CHAP. V.

Concerning the Nature of Æquations, *and how to prepare them for a* Solution.

WHEN any *Problem* or *Queſtion* is propoſed to be *Analytically Reſolved*; it is very requiſite that the true *Deſign* or *Meaning* thereof, be fully and clearly *Comprehended* (in all its parts) that ſo it maybe truly *Abſtracted* from ſuch *Ambiguous Words*, as *Queſtions* of this *kind* are often *Diſguiſed* with; otherwiſe it will be very *Difficult*, if not *Impoſſible*, to *ſtate* the *Queſtion* right in its *ſubſtituted Letters*, and ever to bring it to an *Æquation*, by ſuch *various Methods* of *Ordering* thoſe *Letters* as the *Nature* of the *Queſtions* may *require*.

Now

Now the *Knowledge* of this *difficult Part* of the *Work* is only to be *obtained* by *Practice*, and a *careful* minding the *Solution* of such *Leading Questions* as are in *themselves* very *easy*.

And for that *Reason* I have *inserted* a *Collection* of *seven Questions*; wherein there is *great Variety*.

Having got so *clear* an Understanding of the *Question* proposed as to *place* down all the *Quantities* concerned in their due *Order*, *viz.* all the *substituted Letters*, in such *Order* as their *Nature requires*; the *next* Thing must be to *Consider* whether it be *limited* or not. That is, whether it *admits* of more *Answers* than *One*. And to *discover* that, *observe* the *Two* following *Rules*.

Rule.

When the Number of the Quantities sought, Exceed the Number of the given Æquations, the Question is capable of Innumerable Answers.

Example.

Suppose a *Question* were proposed thus; There are *Three* such *Numbers*, that if the *First* be *Added* to the *Second*, their *Sum* will be 22. And if the *Second* be *Added* to the *Third*, their *Sum* will be 46. What are those *Numbers*.

Let the *Three Numbers* be represented by *Three Letters*, thus call the *First a*, the *Second e*, and the *Third y*.

Then $\left\{ \begin{matrix} a+e=22 \\ e+y=46 \end{matrix} \right\}$ according to the *Question*.

Here the *Number* of *Quantities sought* are *Three*; viz. *a*, *e*, *y*, and the *Number* of the given *Æquations* are but *Two*. Therefore this *Question* is not *limited*, but *Admits* of *various Answers*, because for any *One* of those *Three Letters*, you may take any *Number* at *Pleasure*, that is *Less* than 22. Which with a little *Consideration* will be very *Easy* to *Conceive*.

Rule 2.

When the Number of the given Æquations (not depending upon one another) are just as many as the Number of the Quantities sought; then is the Question truly limited, viz. *each Quantity sought hath but one Single value.*

As for Instance; Let the aforesaid *Question* be *proposed* thus. There are *Three Numbers* (*a*, *e*, and *y*, As before) if the *First* be *Added* to the *Second*, their *Sum* will be 2

he *Second* be *Added* to the *Third*, their *Sum* will be 46; and the *First* be *Added* to the *Third*, their *Sum* will be 36. What e the *Numbers*?

That is, $a+e=22$. $e+y=46$. and $a+y=36$.

ow the *Queſtion* is *perfectly Limited*, each *ſingle Quantity* ha- ng but one *ſingle Value*, to wit $a=6$, $e=16$, and $y=30$.

N. B. *If the Number of the given Æquations* exceeds *the umber of the Quantities ſought*; *they not only* Limit *the Que- on, but oftentimes* render *it impoſſible, by being propos'd In- nſiſtent one to another.*

Having *truly ſtated* the *Queſtion* in its *ſubſtituted Letters*, and nd it *Limited* to one *Anſwer*, (or at *leaſt* ſo bounded as to have certain *determinate Number* of *Anſwers*) then let all thoſe *ſub- tuted Letters* be ſo *ordered* or *compared* together, either by dding, *Subſtracting*, *Multiplying*, Or *Dividing* them, &c. ac- rding as the *Nature* of the *Queſtion requires*, until all the *un- own Quantities* except *One*, are caſt *Off* or *vaniſhed*; but there- great *Care* muſt be taken to *keep* them to an *exact Equality*; d when that *unknown Quantity*, or ſome *Power* of it (as uare, *Cube*, &c.) is found *Equal* to thoſe that are *known*; then e *Queſtion* is ſaid to be brought to an *Equation*, and *conſe- ently* to a *Solution*, viz. *fitted* for an *Anſwer*.

But no *particular Rules* can be *preſcribed* for the *Caſting* off, *getting* away *Quantities* out of an *Æquation*; that *part* of e *Art* is only to be obtained by *Care* and *Practice*. And when at is done, it generally happens ſo, that the *unknown Quan- ty* which is retained in the *Æquation*, is ſo *mixed* and *entang- d* with thoſe that are *known*; that it often *Requires* ſome *rouble* and *Skill* to bring it (or its *Powers*, &c.) to one *Side* of e *Æquation*, and thoſe that are *known* to the other *Side*; (ſtill eeping them to a *juſt Equality*) which the *Ingenious* Man *Scooten* his *Principia Macheſeos Univerſalis*, calls *Reduction* of *Æ- uations*.

The Buſineſs of *Reducing Æquations* (*as of moſt, if not all lgebraick Operations*) is *grounded* and *Depends* upon a right *pplication* of the *Five Axioms* propoſed in Page 146. and there- re, if thoſe *Axioms* be well *Underſtood*, the *Reaſon* of ſuch *perations* muſt needs *Appear* very *plain*, and the *Work* be *eaſily erformed*; as in the following *Sections*.

Sect. 1. Of *Reduction* by Addition.

Reduction by *Addition* is grounded upon *Axiom* 1. and is o the *Transposing* (viz. *the Removing*) of any *Negative Quanti* from either *side* of an *Æquation* to the other *side*, with the *Sig* before it; As in these

Examples.

Suppose	1	$a-b=d$
Then	2	$a=d+b$
For	3	$b=b$
$1+3$	4	$a=d+b$

Again,

Let	1	$aa-d=c-aa$
$1+d$	2	$aa=c-aa+d$
$2+aa$	3	$2aa=c+d$

Let	1	$3a-4=6-a$
$1+\bar{4}$	2	$3a=6+4-a$
$2+a$	3	$4a=6+4=10$

Note, *When any absolute Number Registe'r'd in the Margin, you must dr a Line over it, to distinguish it from other Numbers. As* $\bar{4}$ *in the* 2*d Ste this Example.*

Let	1	$aa-dc-b=dd-2ba$
$1+b$	2	$aa-dc=dd-2ba+b$
$2+dc$	3	$aa=dd-2ba+b+dc$
$3+2ba$	4	$aa+2ba=dd+b+dc$

Suppose	1	$2da-d=cc-3baa-aaa$
$1+aaa$	2	$aaa+2da-d=cc-3baa$
$2+3baa$	3	$aaa+3baa+2da-d=cc$
$3+d$	4	$aaa+3baa+2da=cc+d$, &c.

Sect. 2. Of *Reduction* by Substraction.

Reduction by *Substraction* is grounded upon *Axiom* 2. and perform'd by *Transposing* (or *Removing*) any *Affirmative Quantity* from either *side* of the *Æquation*, to the other *side*, w the *Sign* — before it. As in these

Examples.

Suppose	1	$a+b=d$
And	2	$b=b$
$1-2$	3	$a=d-b$

Let	1	$3a+4=6+a$
$1-a$	2	$2a+4=6$
$2-4$	3	$2a=6-4=2$

Suppose	1	$aa+dc+b=dd+2ba$
$1-2ba$	2	$aa-2ba+dc+b=dd$
$2-dc$	3	$aa-2ba+b=dd-dc$
$3-b$	4	$aa-2ba=dd-dc-b$

Let	1	$aaa+d=cc+3baa+2da$
$-3baa$	2	$aaa-3baa+d=cc+2da$
$-2da$	3	$aaa-3baa-2da+d=cc$
$3-d$	4	$aaa-3baa-2da=cc-d$

Sect. 3. Of *Reduction* by Multiplication.

Fractional Quantities in any *Æquation*, are brought into hole *Quantities*; by *Multiplying* every Term in the *Æquation* with the *Denominators* of the *Fractions*, *per Axiom* 3. in these

Examples.

Suppose	1	$\frac{a}{5}=6$
Then	2	$a=6\times5=30$. For $\frac{a}{5}\times5=\frac{5a}{5}=a$

Let	1	$3a=\frac{dc}{2b}$	Suppose	1	$a=\frac{dd}{a-b}$
$1\times2b$	2	$6ba=dc$	$1\times a-b$	2	$aa-ba=dd$

Suppose	1	$\frac{aa}{b}+c+f=\frac{dx}{a}$
$1\times b$	2	$aa+bc+bf=\frac{dxb}{a}$
$2\times a$	3	$aaa+bca+bfa=dxb$

Suppose	1	$\frac{aaa}{aa-bb}=\frac{ba-bb}{a+b}$
$1\times aa-bb$	2	$aaa=\frac{baaa-bbaa-bbba-bbbb}{a+b}$
$1\times a+b$	3	$aaaa+baaa=baaa-bbaa-bbba+bbbb$

Sect. 4. Of *Reduction* by Division.

When any *Quantity (either known or unknown)* is in every *erm* of an *Æquation*; if the whole *Æquation* be *Divided* by at *Quantity*, it will be *Reduc'd* into *Lower Terms*, *per xiom* 4. As in these

Examples.

Suppoſe	1	$baa+bca=bcd$	Let	1	$aa=7a$
$1\div b$	2	$aa+ca=cd$	$1\div 1a$	2	$a=7$

Let	1	$ffaa+ffcaa-ffa=ffda+ffdda$
$1\div ff$	2	$aa+caa-a=da+dda$
$2\div a$	3	$a+ca-1=d+dd$

Or when the *unknown Quantity* is *Multiplied* (viz. joined with any that is *known*; let the whole *Æquation* be *Divided* by the *known Quantity*, that ſo the *unknown* may be *Cleared*. As in theſe

Examples.

Suppoſe	1	$ba-ca=d$	Let	1	$caa-daa=cd-dd$
$1\div b-c$	2	$a=\frac{d}{b-c}$	$1\div c-d$	2	$aa=\frac{cd-dd}{c-d}=d$

Suppoſe	1	$bbaaa-2bbaa=bda+cba$
$1\div ba$	2	$baa-2ba=d+c$
$2\div b$	3	$aa-2a=\frac{d+c}{b}$

Let	1	$49daa+42aa=7bca+21ca$
$1\div 7$	2	$7daa+6aa=bca+3ca$
$2\div a$	3	$7da+6a=bc+3c$
$3\div$	4	$d=\frac{bc+3c}{7d+6}$

Sect. 5. Of *Reduction* by Involution.

When there happens to be an *Æquation*, between any *Homogeneal* or *like Surds*; Take away the *Radical Signs* from the *Quantities*, and they will become *Rational*. As in theſe

Examples.

Suppoſe	1	$\sqrt{a}=\sqrt{d+c}$	Let	1	$\sqrt[3]{aa}=\sqrt[3]{db+bc}$ } *per Sect.*
$1\odot 2$	2	$a=d+c$	$1\odot 3$	2	$aa=db+bc$ } *4. Ch. 3.*

Or if one *ſide* of the *Æquation* conſiſts of *Surd Quantities*, and the other *ſide* be *Rational*; Then Involve the *Rational Quanti-*

antities to the ſame *Power* (*or height*) with the *Index* of *Surd*, and take away the *Radical Sign*. As in theſe

Examples.

Let 1⊙2	1	$\sqrt{a}=6$	Suppoſe 1⊙2	1	$\sqrt{a}=b+c$
	2	$a=36$		2	$a=bb+2bc+cc$

ppoſe 1⊙3	1	$\sqrt[3]{aa-ba}=d$	Let 1⊙5	1	$\sqrt[5]{aa}=7$
	2	$aa-ba=ddd$		2	$aa=16807$.

Sect. 6. Of *Reduction* by Evolution.

When any *ſingle Powers* of the *Unknown Quantity* is on one *e* of an *Æquation*; *Evolve* both *ſides* of the *Æquation*, accor-ng as the *Index* of that *Power denotes*, and their *Roots* will *Equal*. As in theſe

Examples.

ppoſe 1ww2	1	$aa=36$	Let 1ww3	1	$aaa=27$
	2	$a=\sqrt{36}=6$		2	$a=\sqrt[3]{27}=3$, &c.

ppoſe 1ww2	1	$aa=bb-dd$	Let 1ww3	1	$aaa=b^3+3bbc+3bcc+c^3$
	2	$a=\sqrt{bb-dd}$		2	$a=b+c$

Or if any Compound Power of the unknown Quantity be at ne Side of the *Æquation* (*that hath a true Root of its kind*) volve both Sides of the *Æquation*, and it will be Depreſs'd to Lower Terms. As in theſe

Examples.

uppoſe 1ww2	1	$aa+2ba+bb=dd$	$aa-2ba+bb=ddcc$
	2	$a+b=d$	$a-b=dc$

Here follow a few *Examples* of *Clearing Æquations*, wherein ll the foregoing *Reductions* are promiſcuouſly uſed. As Occaſion requires.

Example 1.

Suppoſe	1	$\frac{aa+c-d}{4}=\frac{g-aa}{b}$	What is $a=$ to
$1\times\frac{-}{4}$	2	$aa+c-d=\frac{4g-4aa}{b}$	

2xb

$2\times b$	3	$baa+bc=bd=4g-4aa$
$3+4aa$	4	$baa+4aa+bc=bd=4g$
$4+bd$	5	$baa+4aa+bc=4g+bd$
$5-bc$	6	$baa+4aa=4g+bd-bc$
$6\div b+4$	7	$aa=\frac{4g+bd-bc}{b+4}$
$7\sqrt{\ }2$	8	$a=\sqrt{\frac{4g+bd-bc}{b+4}}$. As was required.

Example 2.

Suppose	1	$\frac{a+354}{a}=\frac{3a}{354-a}$ What is the Value of a.
$1\times a$	2	$a+354=\frac{3aa}{354-a}$
$2\times 354-a$	3	$125316-aa=3aa$
$3+aa$	4	$4aa=125316$
$4\div 4$	5	$aa=31329$
$5\sqrt{\ }2$	6	$a=\sqrt{31329}=177$ the Value of a requir'd.

Example 3.

Suppose	1	$\sqrt{\frac{aa+3bb}{4}}-\sqrt{\frac{aa-3bb}{4}}=\sqrt{\frac{baa}{c}}:a=?$
$1\odot 2$	2	$\frac{aa+3bb}{4}-2\sqrt{\frac{aa+3bb}{4}}\times\sqrt{\frac{aa-3bb}{4}}:+\frac{aa-3bb}{4}=\frac{baa}{c}$
That is	3	$\frac{aa}{2}-\sqrt{\frac{a^4-9b^4}{4}}=\frac{baa}{c}$
For		$\frac{aa+3bb}{4}+\frac{aa-3bb}{4}=\frac{2aa}{4}=\frac{aa}{2}$
And		$2\sqrt{\frac{aa+2bb}{4}}=\sqrt{\frac{4aa+12bb}{4}}=\sqrt{aa+3bb}$
Then		$\sqrt{aa+2bb}\times\sqrt{\frac{aa-3bb}{4}}=\sqrt{\frac{a^4-9b^4}{4}}$
$3+\sqrt{}$ &c.	4	$\frac{aa}{2}=\frac{baa}{c}+\sqrt{\frac{a^4-9b^4}{4}}$

$4-\frac{baa}{c}$	5	$\frac{aa}{2}-\frac{baa}{c}=\sqrt{\frac{a^4-9b^4}{4}}$
5 ⊙ 2	6	$\frac{a^4}{4}-\frac{ba^4}{c}+\frac{bba^4}{cc}=\frac{a^4-9b^4}{4}$
$+\frac{ba^4}{c}$	7	$\frac{a^4}{4}+\frac{bba^4}{cc}=\frac{a^4-9b4}{4}+\frac{ba^4}{c}$
$7+$	8	$\frac{bba^4}{cc}+\frac{9b^4}{4}=\frac{ba^4}{c}$
$8\div b$	9	$\frac{ba^4}{cc}+\frac{9b^4}{4}=\frac{a^4}{c}$
$9\times cc$	10	$ba^4+\frac{9ccb^3}{4}=ca^4$
10×4	11	$4ba^4+9ccb^3=4ca^4$
$1-4ba^4$	12	$9ccb^3=4ca^4-4ba^4$
$12\div$		$aaaa=\frac{9ccb^3}{4c-4b}$
For	13	$4c-4b\times a^4=4ca^4-4ba^4$
13 ww 2	14	$aa=\sqrt{\frac{9ccb^3}{4-4b}}$
14 ww 2	15	$a=\sqrt{:}\sqrt{\frac{9ccb^3}{4c-4b.}}$ As was required.

By help of these *Reductions (properly applied)* the *unknown Quantity, (a)* or its *Powers*, are *Cleared* and brought to one side of an *Æquation*; and if the *unknown Quantity (a) Chance* to be *Equal* to those that are *known*, the *Question* is *Answered*. As in the first *Example* of *Sect.* 1. and 2.

Or if any *single Power* of the *unknown Quantity (a)* is found *Equal* to those that are known, then the *respective Root* of the *known Quantities*, is the *Answer*; as in the first four *Examples* of *Sect.* 6. &c.

But when the *Powers* of the *unknown Quantity* are either *mixed* with their *Root*; As $aa+ba=dd$, &c. Or do consist of *different Powers*; As $aaa+baa=dd$, &c. Then they are called *Affected*, or *Adfected Æquations*, which require other *Methods* to *Resolve* them, *viz.* to find out the *Value* of (*a*) as shall be shewed further on.

CHAP.

CHAP. VI.

Of Proportional Quantities; both Arithmetical, Geometrical and Musical.

WHAT hath been said of *Numbers* in *Arithmetical Progression*, Chap 6. Part 1. may be *easily applied* to a *Series* of *Homogeneal* or *Like Quantities*.

Sect. 1. Of Quantities in Arithmetical Progression.

Those *Quantities* are said to be in the most *Simple* or *Natural Progression*, that begin their *Series* of *Increase* or *Decrease* with a Cypher:

Thus { 0 : a : $2a$: $3a$: $4a$: $5a$: $6a$: &c. *Increasing.*
0 : $-a$: $-2a$: $-3a$: $-4a$: $-5a$: $-6a$: &c. *Decreasing.*

Or *Universally*, putting a the first *Term* in the *Progression* and e the common *Excess* or *Difference*.

Then { $a : a+e : a+2e : a+3e : a+4e : a+5e : a+6e$: &c.
$a : a-e : a-2e : a-3e : a-4e : a-5e : a-6e$: &c.

In the First of these *Series* it's *evident*, that if there be but *Three Terms*; the *Sum* of the *Extreams* will be *Double* to the *Mean*.

As in these, 0 : a : $2a$: Or, a : $2a$: $3a$: Or, $2a$: $3a$: $4a$, &c. viz. $2a+0=a+a$: Or, $a+3a=2a+2a$, &c.

Also, in the Second Series, either Increasing or Decreasing it's *evident*, that if the *Terms* be $a : a+e : a+2e$, &c. *increasing*; Then $a+a+2e$, *viz.* $2a+2e$ the *Sum* of the *Extreams*, is *double* to $a+e$ the *Mean*; Or if they be $a : a-e : a-2e$, &c. *decreasing*; Then $a+a-2e$: *viz.* $2a-2e$ the *Sum* of the *Extreams*, is *double* to $a-e$ the *Mean*. And so it will be in any other *Three* of the *Terms*. *Secondly*, If there are *Four Terms*; then the *Sum* of the *Two Extreams*, will be *Equal* to the *Sum* of the *Two Means*. As in these, $a : a+e : a+2e : a+3e$, in the Series *increasing*; Here $a+a+3e=a+e+a+2e$.

Also in these, $a : a-e : a-2e : a-3e$ in the Series *decreasing*; here $a+a-3e=a-e+a-2e$, &c. in any other Four *Terms*.

Consequently, If there are never so many *Terms* in the Series the Sum of the *Two Extreams* will always be *Equal* to the Sum

ny *Two Means*, that are *equally distant* from those *Extreams*: in these, $a : a+e : a+2e : a+3e : a+4e : a+5e$: &c. re $a+a+5e = a+e+a+4e = a+2e+a+3e$, &c. d if the *Number* of *Terms* be *Odd*; the *Sum* of the *Two treams* will be *double* to the *Middle Term*, &c. As in *Corol.* 1. *ap.* 6. before-mentioned.

Consectary 1.

Whence it follows, (and is very easy to conceive) that if the m of the Two Extreams be multiplied into the Number of all Terms in the Series, the Product will be Double the Sum of the Series.

Now for the easier resolving such Questions as depend upon these ogressional Quantities,

t $\begin{cases} a = \text{the } \textit{First Term, as before.} \\ y = \text{the } \textit{Last Term.} \\ e = \text{the } \textit{Common Excess}, \text{ \&c. } \textit{as before.} \\ N = \text{the } \textit{Number of all the Terms.} \\ S = \text{the } \textit{Sum of all the Series}, \text{ viz. } \textit{of all the Terms.} \end{cases}$

Then will $a+y \times N = 2S$, by the *precedent Consectary*:

That is, $Na+Ny = 2S$. *Consequently* $\frac{Na+Ny}{2} = S$, the

m of all the Series, be the *Terms* never so many. *Thirdly*, In ese Series, it is easy to *perceive*, that the *Common Difference*) is so often *Added* to the *Last Term* of the Series; as are the *umber* of *Terms*, except the *First*; That is, the *First Term (a)* th no *Difference Added* to it, but the *Last Term* hath so many nes *(e) Added* to it, as it is distant from the *First*.

Consequently, the *Difference* betwixt the *Two Extreams*, is ly the *Common Difference (e) Multiplied* into the *Number* of the *Terms Less Unity* or 1.

That is, $N-1 \times e = y-a$, the *Difference* betwixt the wo *Extreams*, *viz.*: $Ne-e = y-a$.

Consectary 2.

Whence it follows, that if the Difference betwixt the Two Exeams be Divided by the Number of Terms Less 1. *the Quotient ill be the Common Difference of the Series.*

To wit, $\frac{y-a}{N-1} = e$.

Now by the Help of these *Two Consectaries*, if any *Three* the aforesaid *Five Parts (viz. a. y. e. N. S.)* be given; the oth *Two* may be *easily* found.

Thus,	1	$\frac{Na+Ny}{2}=S$	*As before.*
And	2	$\frac{y-a=e}{N-1}$	*As before.*
$2\times N-1$	3	$y-a=Ne-e$	
$3+e$	4	$y-a+e=Ne$	
$4\div e$	5	$\frac{y-a+e}{e}=N$	*The Number of Terms.*
1×2	6	$Na+Ny=2S$	
$6-Na$	7	$Ny=2-Na$	
$7\div N$	8	$\frac{2S-Na}{N}=y$	*The last Term*
$6-yN$	9	$Na=2S-Ny$	
$9\div N$	10	$\frac{2S-Ny}{N}=a$	*The first Term*
$6\div a+y$	11	$\frac{2S}{a+y}=N$	*The Number of Terms.*
5, and 11	12	$\frac{y-a+e}{e}=\frac{2S}{a+y}$	*per Axiom* 5.
$12\times a+y$	13	$\frac{yy-aa}{e}+a+y=2S$	
$13\div 2$	14	$\frac{yy-aa}{2e}+\frac{a+y}{2}=S$	*The* Sum *of all the* Seri
$14\times 2e$	15	$yy-aa+ae+ye=2Se$	
$15-ae$	16	$yy-aa+ye=2Se-ae$	
$16-ye$	17	$yy-aa=2Se-ae-ye$	
$17\div$	18	$\frac{yy-aa}{2S-a-y}=e$	*The Common Difference.*
$3+a$	19	$Ne-e+a=y$	*The last Term*
$19+e$	20	$Ne+a=y+e$	
$20-Ne$	21	$y+e-Ne=a$	*The first Term.*
			&c.

In like Manner you may proceed to find out any of the Fi *Quantities (a. e. y. N. S.)* otherways, *viz.* by *Varying* a *Comparing* of these *Æquations* one with another, you ma *produce* new *Æquations* with other *Data* in them; the whic I sha

hall here omit *purſuing*, and *Leave* them for the *Learner's* ractice.

Sect. 2. Of Quantities in *Geometrical Proportion.*

Geometrical Proportion continued has been already Defined in ct. 2. *Chap.* 6. *Part* 1. And what is there ſaid concerning umbers in ∺ may eaſily be applied to any Sort of *Homogeneal uantities* that are in ∺.

The moſt *Natural* and *Simple Series* of *Geometrical Proporals*, is when it begins with *Unity* or 1.

As $1 . a . aa . aaa . aaaa . a^5 . a^6$, &c. in ∺

For $1 : a :: a : aa :: aa : aaa :: aaa : aaaa$, &c.

Or $a . b . \frac{bb}{a} . \frac{bbb}{aa} . \frac{bbbb}{aaa} . \frac{b^5}{a^4}$, &c. are *Terms* in ∺

For $a : b :: b : \frac{bb}{a} :: \frac{bb}{a} : \frac{bbb}{aa} :: \frac{bbb}{aa} : \frac{b^4}{a^3} :: \frac{b^4}{a^3} : \frac{b^5}{a^4}$. &c.

That is, when all the *middle Terms* betwixt the *Two Extreams* e both *Conſequents* and *Antecedents*, that Series is in *Geomeical Proportion continued*.

Therefore in every *Series* of *Quantities* in ∺ all the *Terms cept* the *Laſt* are *Antecedents*; and all the *Terms except* the *rſt* are *Conſequents*.

But *Univerſally* putting *a* the *Firſt Term* in the Series, d *e* the *Ratio*, *viz.* the *Common Multiplier*, or *Diviſor*, then will be

$a . ae . aee . aeee . aeeee . ae^5\ ae^6$. &c. in ∺

Or $a . \frac{a}{e} . \frac{a}{ee} . \frac{a}{eee} . \frac{a}{eeee} . \frac{a}{e^5}$ &c. are in ∺ Decr.

For $a : ae :: ae : \frac{aaee}{a} = aee$, &c.

And $a : \frac{a}{e} :: \frac{a}{e} : \frac{aa}{aee} = \frac{a}{ee}\ a : \frac{a}{e} :: \frac{a}{ee} : \frac{a}{eee}$ &c.

1. In any of theſe Series it is *evident*, that if *Three Quanies* are in ∺ the *Rectangle* of the *Two Extreams* will be *ual* to the *Square* of the *Mean*.

As in theſe. $a : ae . aee$ here $a \times aee = ae \times ae. = aaee$. &c.

Or $a \cdot \frac{a}{e} \cdot \frac{a}{ee}$ here alſo, $a \times \frac{a}{ee} = \frac{a}{e} \times \frac{a}{e} = \frac{aa}{ee}$ &c.

II. If *Four Quantities* are in ∺ the *Rectangle* of the *Extreams* will be *Equal* to the *Rectangle* of the *Means*.

As in theſe, $a \cdot ae \cdot aee \cdot aeee$. here $a \times ae^3 = ae \times aee$.

Or $a \cdot \frac{a}{e} \cdot \frac{a}{ee} \cdot \frac{a}{eee}$ here alſo $a \times \frac{a}{eee} = \frac{a}{e} \times \frac{a}{ee} = \frac{aa}{eee}$ &c.

Conſequently, If there are never ſo many *Terms* in the *Series* of ∺ the *Rectangle* of the *Extreams* will be Equal to the *Rectangle* of any *Two Means* that are *Equally diſtant* from the *Extreams*.

As in theſe, $a \cdot ae \cdot aee \cdot aeee \cdot aeeee \cdot ae^5$ &c.
viz. $ae^5 \times a = ae^4 \times ae$. Or $ae^5 \times a = aeee \times aee = aae^5$.

III. If never ſo many Quantities are in ∺ it will be, As any *one* of the *Antecedents* is to its *Conſequents*; So is the *Sum* of all the *Antecedents*, to the *Sum* of all the *Conſequents*.

As in theſe, $\begin{cases} a \cdot ae \cdot aee \cdot aeee \cdot aeeee \cdot ae^5, \text{ &c. Increaſing.} \\ a \cdot \frac{a}{e} \cdot \frac{a}{ee} \cdot \frac{a}{eee} \cdot \frac{a}{eeee} \cdot \frac{a}{e^5} \text{ &c. Decreaſing.} \end{cases}$

$a : ae :: a + ae + aee + ae^3 + ae^4 : ae + aee + ae^3 + ae^4 + a$

Or $a : \frac{a}{e} :: + \frac{a}{e} + \frac{a}{ee} + \frac{a}{e^3} + \frac{a}{e^4} \quad \frac{a}{e} + \frac{a}{e} + \frac{a}{ee} + \frac{a}{e^3} + \frac{a}{e^4} + \frac{a}{e^5}$

viz. $a \times ae + aee + ae^3 + ae^4 + ae^5 = ae \times a + ae + aee + ae^3 + a$

That is, the *Rectangle* of the *Extreams* is *Equal* to the *Rectangle* of the *Means*; *per* Second of this *Sect.*

Note, The *Ratio* of any *Series* in ∺ *increaſing* is found by *Dividing* any of the *Conſequents* by its *Antecedents*.

Thus, $a) ae (e$ Or $ae) aee (e$ &c.

But if the *Series* be *Decreaſing*, then the *Ratio* is found by *Dividing* any of the *Antecedents* by its *Conſequents*.

Thus, $\frac{a}{e}) a (e$ Or $\frac{a}{ee}) \frac{a}{e} (e$ &c.

Co

Conſectary.

Theſe Things being premiſed, ſuch Æquations may be deduced om them, as will ſolve all ſuch Queſtions as are uſually propoſed out Quantities in Geometrical Proportion ÷ In order to that

Let $\left\{\begin{array}{l} a = \text{The } \textit{First Term.} \\ e = \text{The } \textit{Common Ratio.} \\ y = \text{The } \textit{Laſt Term.} \\ S = \text{The } \textit{Sum} \text{ of all the } \textit{Terms.} \end{array}\right\}$ as before.

Then $S-y$ = the *Sum* of all the *Antecedents.*
And $S-a$ = the *Sum* of all the *Conſequents.*

Analogy.	1	$a : ae :: S-y : S-a$ *Per* the III. of this *Sect.*
1 ∴	2	$Sa-aa=aeS-aey$
$2 \div a$	3	$S-a=eS-ey$
$3+ey$	4	$S+ey-a=eS$
$4-S$	5	$ey-a=eS-S$
$5 \div e-1$	6	$\frac{ye-a}{e-1}=S$ The *Sum* of all the *Series.*
$3 \div S-y$	7	$\frac{S-a}{S-y}=e$ The *common Ratio.*
$5+a$	8	$ey=eS+a-S$
$8 \div e$	9	$\frac{eS+a-S}{e}=y$ The *Laſt Term.*
$4+a$	10	$S+ey=eS+a$
$10-eS$	11	$S+ey-eS=a$ The *First Term.*

Note, The ∴ ſet in the *Margin* at the *ſecond Step*, is inſtead f *Ergo*; and imports that the *Rectangle* of the *Two Extreams* n the Firſt *Step*, is *Equal* to the *Rectangle* of the *Means.* And o for any other *Proportion.*

Sect 3. Of Harmonical *Proportion.*

Harmonical or *Muſical Proportion* is, when of *Three Quantities (or rather Numbers)* the *Firſt* hath the ſame *Ratio* to the *Third*, As the *Difference* between the *Firſt* and *Second*, hath to he *Difference* between the *Second* and *Third.* As in theſe following.

Suppoſe *a*, *b*, *c* in *Muſical Proportion.*

Then	1	$a : c :: b-a : c-b$
1 ∴	2	$cb-ca=ac-ba$

2+

$2+ca$	3	$cb=2ac-ba$
$3\div 2c-b$	5	$\frac{cb}{2c-b}=a$ The *First Term.*
$3+ba$	4	$2ac=cb+ba$
$5\div c+a$	6	$\frac{2ac}{c+a}=b$ The *Second Term.*
$5-cb$	7	$2ac-cb=ba$
$7\div 2a-c$	8	$\frac{ba}{2a-b}=c$ The *Third Term.*

If there are *Four Terms* in *Musical Proportion*, the *First* hath the same *Ratio* to the *Fourth*, as the *Difference* between the *First* and *Second* hath to the *Difference* between the *Third* and *Fourth*.

That is, Let *a. b. c. d* be the *Four Terms*, &c.

Then	1	$a : d :: b-a : d-c$
$1\div$	2	$db-da=da-ca$
$2+da$	3	$db=2da-ca$
$3\div 2d-c$	4	$\frac{db}{2d-c}=a$
$3\div 4$	5	$b=\frac{2da-ca}{d}$
$3\times ca$	6	$db+ca=2da$
$6-db$	7	$ca=2da-db$
$7\div a$	8	$c=\frac{2da-db}{a}$
$7\div 2a-b$	9	$\frac{ca}{2a-b}=d$

CHAP. VII.

Of Proportion **Disjunct**, *and how to* turn Æquations *into* **Analogies**, &c.

Proportion Disjunct, or the *Rule of Three* in *Numbers*, is already explain'd in *Chap.* 7. *Part* 1. And what hath been there said, is applicable to all Homogeneous Quantities, *viz.* of Lines to Lines, *&c.*

Sect. 1.

If *Four Quantities*, (viz. *either Lines*, *Superficies*, or *Solids*) be *Proportional*: the *Rectangle* comprehended under the *Extreams*

equal to the *Rectangle* comprehended under the *Two Means*, *Euclid 6.)*

For Inſtance, Suppoſe, $a . b . c . d .$ to repreſent the *Four mogeneal Quantities* in *Proportion.*

viz. $a : b :: c : d.$ Then will $ad = bc.$

For ſuppoſe $b = 2a$ then will $d = 2c$

And it will be $a : 2a :: c : 2c.$ Here the *Ratio* is 2.

but $a \times 2c = 2a \times c.$ *viz.* $2ca = 2ac$

Or ſuppoſe $b = 3a$ then will $d = 3c$

And it will be $a : 3a :: c : 3c.$ Here the *Ratio* is 3.

but $a \times 3c = 3a \times c.$ *viz.* $3ca = 3ac.$

Or *Univerſally* putting e for the *Ratio* of the *Proportion*, making $b = ae$, then will $d = ce$

And it will be $a : ae :: c : ce$

but $a \times ce = ae \times c$ viz. $ace = aec.$

Conſequently, $ad = bc$ which was to be *proved.*

Whence it follows, that if any *Three* of the *Four Proportional uantities* be given, the *Fourth* may be eaſily found.

Thus,

Let	1	$a : b :: c : d.$ Then
1 ∴	2	$ad = bc$ as before
$2 \div d$	3	$a = \frac{bc}{d}$
$2 \div c$	4	$b = \frac{ad}{c}$
$2 \div b$	5	$c = \frac{ad}{b}$
$2 \div a$	6	$d = \frac{bc}{a}$
$2 \div bd$	7	$\frac{a}{b} = \frac{c}{d}$
$2 \div ac$	8	$\frac{b}{a} = \frac{d}{c}$

Note, *In this Manner* Euclid, *in his 5th Book expreſſes the Ratio of Proportionals*, viz. *the Ratio of* a *to* b *is* $\frac{a}{b}$

If *Four Quantities* are *Proportional*, they will alſo be *Pro- tional* in *Alternation*, *Inverſion*, *Compoſition*, *Diviſion*, *Con- ſion*, and *Mixtly*. *Euclid* 5. *Def.* 12, 13, 14, 15, 16.

That

That is, if	1	$a : b :: c : d$ be *in Direct Proportion*, as before
Then	2	$a : c :: b : d$ *Alternate*. For $ad = bc$
And	3	$b : a :: d : c$ *Inverted*. For $ad = bc$
Alſo	4	$a+b : b :: c+d : d$ *Compounded*.
4 ∴	5	$da+bd = bc+bd$ That is, $ad = bc$, as before.
Or	6	$a+c : c :: b+d : d$ *Alternately Compounded*.
6 ∴	7	$ad+cd = bc+cd$ That is, $ad = bc$.
Again,	8	$a-b : b :: c-d : d$ *Divided*.
8 ∴	9	$ad-bd = bc-bd$ That is, $ad = bc$.
Or	10	$a-c : c :: b-d : d$ *Alternately Divided*.
10 ∴	11	$ad-cd = bc-cd$ That is, $ad = bc$.
And	12	$a : b \pm a :: c : d \pm c$ *Converted*
12 ∴	13	$ad \pm ac = bc \pm ac$ That is, $ad = bc$.
Laſtly	14	$a+b : a-b :: c+d : c-d$ *Mixtly*.
14 ∴	15	$ac-ad+bc-bd = ac+ad-bc-bd$
15 $\pm$	16	$2bc = 2ad$; That is, $ad = bc$ As at firſt.

Note, What has been here done about whole *Quantities* in *Simple Proportion*, may be eaſily perform'd in *Fractional Quantities* ; And *Surds*, &c.

For Inſtance, If $\frac{ab}{c} : \frac{d-c}{f} :: \frac{d+c}{c}$ and it be required to find the *Fourth Term*,

It will be $\frac{dd-cc}{fc}$ the Rectangle of the Means ; which being

Divided by the Firſt *Extream* $\frac{ab}{c}$ it will become

$$\frac{ab}{c}\Big)\frac{dd-cc}{fc}\Big(\frac{ddc-ccc}{abfc} = \frac{dd-cc}{abf}$$ the *Fourth Term*.

Or if $b : \sqrt{bd+bc} :: \sqrt{bd+bc}$: to a *Fourth Term*.
Then is, $\sqrt{bd+bc} \times \sqrt{bd+bc} = bd+bc$ the *Rectangle* of the *Means*, And $b)\ bd+bc\ (d+c$ the *Fourth Term*.
That is, $b : \sqrt{bd+bc} :: \sqrt{bd+bc} : d+c$ &c.

Sect. 2. Of Duplicate and Triplicate Proportion.

The *Proportions* treated of in the laſt *Section*, are to be underſtood when *Lines* are compared to *Lines*, and *Superficies* to *Su*,

perficies; or *Solids* to *Solids*; *viz.* when each is compared to t of its like kind, which is only called Simple *Proportion*.
But when *Lines* are compared to *Superficies*, or *Lines* are com-ed to *Solids*, such Comparisons are distinguished from the for-r, by the Names of *Duplicate* and *Triplicate*, (&c.) *Propor-ns*; so that *Simple*, *Duplicate* and *Triplicate*, &c. *Proporti-* are to be understood in a different Sense from *Single*, *Double*, *ble*, &c. *Proportions*, which are only as 1, 2, 3, &c. to 1; those of *Simple*, *Duplicate*, *Triplicate*, &c. *Proportions*, is t of a . aa . aaa ., &c. to 1. Or if the *Simple Proportions* that of a to b, whose Ratio or Exponent is $\frac{a}{b}$ or $\frac{b}{a}$ ording to *Euclid*'s Way.

en $\frac{a}{b} \times \frac{a}{b} = \frac{aa}{bb}$ is the Exponent of the Duplicate

d $\frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a^3}{b^3}$ is the Exponent of the Tr.Prop. } of $\frac{a}{b}$

&c.

And if there are *Three*, *Four*, or more *Quantities*, in $\div$ as a . aa . aaa . a^4 . a^5, &c. (As in the First *Series*, *Sect.* 2. of last Chapter.) Then, that of the First to the Third, Fourth Fifth, &c. (*viz.* 1 To aa . aaa . a^4 . a^5) is *Duplicate*, *Tripli-*, *Quadruplicate*, &c. of the First to the Second, (*viz.* of 1 a;) And by *Inversion*, that of the Third, Fourth, Fifth, is *plicate*, *Triplicate*, &c. of that of the *Second* to the *First* To 1) *per def.* 10. *Eucl.* 5. But the Nature of these *Propor-s* will appear more *Evident*, and be easier understood when are Applied to Practice, and illustrated by *Geometrical Fi-es*. Further on.

Sect. 3. How to *turn Æquations* into Analogies.

rom the first *Section* of this *Chap.* it will be easy to conceive to turn or dissolve *Æquations* into *Analogies* or *Proportions*.
or if the *Rectangle* of Two (or more) *Quantities*, be Equal e *Rectangle* of Two (or more) *Quantities*; then are those (or more) *Quantities Proportional*. By the 16 *Eucl.* 6.
hat is, if $ab = dc$. Then is $a : c :: d : b$.
Or $c : a :: b : d$ &c.
rom whence there arise this general *Rule* for turning *Æqua-* into *Analogies*.

Rule.

Divide either Side of the given Æquation, (if it can be do into Two ſuch Parts, or Factors, as being multiplied toget will produce that Side again; and make thoſe Two Parts Two Extreams. Then Divide the other Side of the Æquat (if it can be done) in the ſame Manner as the Firſt was, let thoſe Two Parts or Factors, be the Two Means.

For Inſtance, Suppoſe $ab+ad+bd$.
Then $a:b::d:b+d$. Or $b:a::b+d:d$ &c.
Or taking ad from both Sides of the Æquations, and
It will be $ab=bd-ad$. Then $a:d::b-a:b$.
Or, $b:d::b-a:a$ &c.
Again, ſuppoſe $aa+2ae=2by+yy$.
Here a and $a+2e$ are the Two Factors of the Firſt Side in t Æquation; for $a+2e\times a=aa+2ae$.
Again, y and $2b+y$ are the Two Factors of the other ſide
Therefore, $a:y::2b+y:a+2$
Or $2b+y:a+2e::a:y$ &c.

When one Side of any Æquation can be divided into T Factors, as before; and the other Side cannot be ſo Divid then make the Square Root of that Side either the Two *treams* or the Two *Means*.

For inſtance, Suppoſe $bc+bd=da+g$.
Then $b:\sqrt{da+g}::\sqrt{d+g}:c+d$
Or $\sqrt{da+g}:b::c+d:\sqrt{da+g}$. &c.

CHAP. VIII.

Of Subſtitution, and the *Solution* of Quadratick Æquatio

Sect. 1. Of Subſtitution.

When new *Quantities* not concerned in the Firſt ſtating of *Queſtion*, are put inſtead of ſome that are engaged in it, th called *Subſtitution*.

For inſtance, If inſtead of $\sqrt{bc-dc}$ you put z, or any o Letter.

That is, make $z=\sqrt{bc-dc}$.

Or Suppoſe $aa+ba-ca+da=dc$, inſtead of $b-c+d$ put s, any other Letter not engaged in the *Queſtion*, Viz. $s=b-c+d$. Then $aa+sa=dc$.
That is, if c be greater than $b+d$, it's $aa+sa=dc$.
But if $b+d$ be greater than c, Then it's $aa+sa=dc$.
And this way of *Subſtituting* or putting of new *Quantities* ſtead of others, may be found very uſeful upon ſeveral Occaons; *viz.* in order to make ſome following *Operations* in the *ueſtion* more eaſy, and perhaps much ſhorter than they would without it, as you may obſerve in ſome *Queſtions* hereafter poſed in this *Tract*.
And when thoſe *Operations*, in which the *Subſtituted Quantis* were aſſiſting or uſeful, are performed according as the Nare of the *Queſtion* required, you may then (if there be Ocſion) bring the *Original* or Firſt *Quantities* into the *Æquati*, in the Place (or Places) of thoſe *Subſtituted Quantities*, hich is called *Reſtitution*, as you may ſee further on.

Sect. 2. The *Solution* of Quadratick Æquations.

When the *Quantity* ſought is brought to an *Equality* with oſe that are known, and is on one Side of the *Æquation*, in no ore than Two Different *Powers* whoſe *Indices* are double one another, thoſe *Æquations* are called *Quadratick Æquations dfected*; and do fall under the Conſideration of Three *Forms Caſes*.

Caſe 1. $aa+2ba=dc.$
Caſe 2. $aa-2ba=dc.$
Caſe 3. $2ba-aa=dc.$
And $\begin{cases} a^4+2ba^2=dc. \\ a^4-2ba^2=dc. \\ 2ba^2-a^4=dc. \end{cases}$

Alſo $\begin{cases} a^6+2ba^3=dc. \\ a^6-2ba^3=dc. \\ 2ba^3-a^6=dc. \end{cases}$ And $\begin{cases} a^8+2ba^4=dc. \\ a^8-2ba^4=dc. \\ 2ba^4-a^8=dc. \end{cases}$ &c.

When there happens to be more *Terms* in one of theſe kind *Æquations* than Two, and the *higheſt Power* of the unknown *uantity* is *Multiplied* into ſome known *Co-efficients*; you muſt educe them by *Diviſion*; as in *Sect.* 4. of *Chap.* 5. and for the *actional Quantities* that may ariſe by thoſe Diviſions, *Subſtiute* another *Quantity* Doubled.

For Inſtance, Let $baa+caa-ca-da=dc+cb$.

Then $aa-\frac{ca-da}{b+c}=\frac{dc+cb}{b+c}$ make $\frac{c-d}{b+c}=2x$.

And if you pleaſe, for $\frac{dc+cb}{b+c}$ put z.

Then will $aa-2xa=z$ be the new *Æquation*, equal to the other, being now fitted for a *Solution*.

Now any of theſe Three *Forms* of *Æquations* being thus prepared for a *Solution*, may be reduced to ſimple *Powers* by caſting off the Second or *Loweſt Term* of the unknown *Quantity*, which is done by *Subſtitution*, thus, always take *half* the known *Co-efficient*, and Add it to (Caſe 1.) or Subſtract it from (Caſe 2.) its Fellow *Factor*; and for their *Sum*, or *Difference*, *Subſtitute* another *Letter*. As in theſe.

Let	1	$aa+2ba=dc$ Caſe 1.
Put	2	$a+b=e$
2 ⊙ 2	3	$aa+2ba+bb=ee$
3 1	4	$bb=ee-dc$
4 + dc	5	$ee=bb+dc$
5 ω 2	6	$e=\sqrt{bb+dc}$
2 and 6	7	$a+b=\sqrt{bb+dc}$ *Per* Axiom 5.
7 − b	8	$a=\sqrt{bb+dc}\,;-b$

Again.

Let	1	$aa-2ba=dc$ Caſe 2.
Put	2	$a-b=e$
2 ⊙ 2	3	$aa-2ba+bb=ee$
3 − 1	4	$bb=ee-dc$
4 + dc	5	$ee=dc+bb$
5 ω 2	6	$e=\sqrt{dc+bb}$
2, 6	7	$a-b=\sqrt{dc+bb}$
7 + b	8	$a=b+\sqrt{dc+bb}$

In Caſe 3. From half the known *Co-efficient* Subſtract its fellow Factor.

Thus, Let	1	$2ba-aa=dc$
Put	2	$b-a=e$
2 ⊙ 2	3	$bb-2ba+aa=ee$
1 + 3	4	$bb=dc+ee$
4 − dc	5	$ee=bb-dc$

5 ω

5 ww 2	6	$e=\sqrt{bb-dc}$
(2, 6	7	$b-a=\sqrt{bb-dc}$
7+a	8	$b=a+\sqrt{bb-dc}$
√, &c.	9	$a=b-\sqrt{bb-dc}$

And this *Method* holds good in thoſe other *Æquations*, where-he *higheſt Powers* are a^4, a^6, a^8, &c. As for inſtance.

Let	1	$a^6+2ba^3=dc$. Caſe 1.
Put	2	$a^3+b=e$
2 ⊙ 2	3	$a^6+2ba^3+bb=ee$
3 — 1	4	$bb=ee-dc$
4+dc	5	$ee=bb+dc$
5 ww 2	6	$e=\sqrt{bb+dc}$
2, 6	7	$a^3+b=\sqrt{bb+dc}$
7—b	8	$a^3=\sqrt{bb+dc}:-b$
8 ww 3	9	$a=\sqrt[3]{:\sqrt{bb+dc}:-b}$

The ſame may be done with all the *reſt*, Care being taken to *d*, *or Subſtract*, according as the Caſe *requires*.

But all *Quadratick Æquations* may be more *eaſily Reſolved* *Compleating* the *Square* which is grounded upon the *Conſide-ion* of *Raiſing* a *Square* from any *Binomial*, or *Reſidual* ot, (See *Sect.* 5. *Chap.* 1.)

Viz. If $a+b$ be *involved* to a *Square*, it will be $aa+2ba+bb$

And if $a-b$ be ſo involved, it will be $aa-2ba+bb$

Whence it is *eaſy* to *obſerve*, that $aa+2ba=dc$. Caſe 1.

And $aa-2ba=dc$. Caſe 2. Are *imperfect Squares*, wanting *y* bb to make them *Compleat*. And therefore it is, that if *f* the known *Co-efficient* be *involved* to the *Second Power*, the *Square* be *Added* to both Sides of the *Æquation*, the *known Side* will become a *Compleat Square*.

us, Let	1	$aa+2ba=dc$.	Here *half* the *Co-efficient* $2b$ is b, which being *ſquared*, is bb.
But	2	$bb=bb$	
1+2	3	$aa+2ba+bb=dc+bb$	Caſe 1.
3 ww 2	4	$a+b=\sqrt{dc+bb}$	As before.

Again.

Again.

Let	1	$aa-2ba=dc$ Caſe 2.
But	2	$bb=bb$
1+3	3	$aa-2ba+bb=dc+bb$
3ωω2	4	$a-b=\sqrt{dc+bb}$ &c. As before.

But in Caſe 3. you muſt change the *Signs* of all the *Terms* in the *Æquation*,

Thus	1	$2ba-aa=dc$ Caſe 3.
1±	2	$aa-2ba=-dc$
Then	3	$aa-2ba+bb=bb-dc$ &c.

And this *Method* of *Compleating* the *Square*, holds true in the other *Æquations*.

Viz.	1	$aaaa+2baa=dc$ Caſe 1.
For	2	$bb=bb$ As before.
1+2	3	$aaaa+2baa+bb=dc+bb$
3ωω2	4	$aa+b=\sqrt{dc+bb}$
4−*b*	5	$aa=\sqrt{dc+bb}:-b$
5ωω2	6	$a=\sqrt{:\sqrt{dc+bb}:-b}$ And ſo on for the reſt.

Or let	1	$a^6+2baaa=dc$ As before, Caſe 1.
And	2	$bb=bb$
1+2	3	$a^6+2baaa+bb=dc+bb$
1ωω2	4	$aaa+b=\sqrt{dc+bb}$
4−*b*	5	$aaa=\sqrt{dc+bb}:-b$
5ωω3	6	$a=\sqrt[3]{:\sqrt{dc+bb}:-b}$ &c.

Corollary.

Hence it's evident, that whatſoever Method is uſed in ſolving theſe (or indeed any other) Æquations, the Reſult will ſtill be the ſame, if the Work be true; as you may obſerve from the Operations of this Section: For both theſe Methods here propoſed, give the ſame Theorems in their reſpective Caſes for the Value of (a).

Thus

Thus, when $aa+2ba=dc$ Then
Theorem 1. $a=\sqrt{dc+bb}:-b$
And when $aa-2ba=dc$ Then
Theorem 2. $a=b+\sqrt{dc+bb}$
Again, when $2ba-aa=dc$ Then
Theorem 3. $a=b-\sqrt{bb-dc}$

The *like Theorems* may be eaſily raiſed for the reſt.

If the known *Co-efficients (of the Second or Loweſt Term)* be by *ſingle Quantity*, as $aa+ba=dc$, &c. Then is $\frac{1}{2}b$ its *Half*, and $\frac{1}{4}bb$ will be the Square of that *Half*: That is, $\frac{1}{2}b\times\frac{1}{2}b=\frac{1}{4}bb$. And then the Work will ſtand

Thus,	1	$aa+ba=dc$
1 *C* □	2	$aa+ba+\frac{1}{4}bb=dc+\frac{1}{4}bb$
2 *ω* 2	3	$a+\frac{1}{2}b=\sqrt{dc+\frac{1}{4}bb}$
$3-\frac{1}{2}b$	4	$a=\sqrt{dc+\frac{1}{4}bb}:-\frac{1}{2}b$. And ſo for the reſt.

Note, *C* □ *placed in the Margin againſt the Second Step, ſignifies that the imperfect* Square $aa+ba$ *in the firſt* Step, *is here compleated*, viz. *in the ſecond* Step.

Now by the Help of theſe *Theorems*, it will be *eaſy* to *Calculate* or find the *Value* of the *unknown Quantity*, *(a)* in *Numbers*.

Example 1.

Suppoſe $aa+2ba=z$. Let $b=16$. And $z=4644$.
Then $a=\sqrt{z+bb}:-b$ *per* Theorem 1.
But $z+bb=4644+256=4900$ And $\sqrt{4900}=70$
Conſequently $a=70-16$, viz. $a=54$.

But every *Adfected Æquation*, hath as many *Roots (or rather Values* of the *unknown Quantity)* either *Real* or *Imaginary*, as are the *Dimenſions* (viz. *the Index)* of its *higheſt Power*; and therefore the *Quantity a*, in this *Æquation*, hath another *Value* either *Affirmative* or *Negative*; which may be thus found.

The given *Æquation* is $aa+32a=4644$, and its *Root* $a=54$.

Let theſe two *Æquations* be made *Equal* or *Æquated* to 0, viz. to Nothing.

Thus,

Thus, $aa+32a-4644=0$ And $a-54=0$.
Then *Divide* the given *Æquation* by its *first Root*, and the *Quotient* will *shew* the *second Value* of a.

Thus, $a-54=0)\ aa+32a-4644=0\ (a+86=0$
$aa-54a$
$+86a-4644$
$86a-4644$
(0)

Hence the *Second Value* of a is $=-86$, Or $86=-$ which *seems impossible*, *viz.* that an *Affirmative Quantity* should be *Equal* to a *Negative Quantity*; yet even by this *Second Value* of a, and the same *Co-efficient*, the true *(or first) Equation* may be formed.

Thus, Let	1	$a=-86$
$1 \odot 2$	2	$aa=+7396$, viz. $-86\times-86=+7396$
1×32	3	$32a=-2752$
$2+3$	4	$aa+32a=4644$. As at first.

Example 2.

Suppose	1	$aa-7a=948,75$ Then *per Theorem* 2.
$1\ C\ \square$	2	$aa-7a+\frac{49}{4}=948,75+\frac{49}{4}=961$
$2\ ww\ \frac{2}{}$	3	$a-\frac{7}{2}$ (Or 3,5) $=\sqrt{961}=31$
$3+35$	4	$a=31+3,5=34,5$

Again, for the Second Value of a,
Let $aa-7a-948,75=0$. And $a-34,5=0$
Then, $a-34,5=0)\ aa-7a-948,75=0\ (a+27,5=0$
Consequently this second *Value* is $a=-27,5$
which will *form* the *Original Æquation*, $aa-7a=948,75$ if it be ordered as the last was.

Example 3.

Suppose $36a-aa=243$ Then *per Theorem* 3.
$a=18-\sqrt{324-243}$ *viz.* half 36 *squared* is 324 &c.
That is, $a=18-\sqrt{81}$ but $\sqrt{81}=9$.
Therefore $a=18-9=9$. Now this *Third Form* is called an *Ambiguous Æquation*, because it hath *Two Affirmative* Values of the unknown *Quantity (a)*, both which may be found without such *Division* as was used before.

Jo

For in this Case, $a = 18 + \sqrt{81}$, viz. $a = 18 + 9 = 27$, $a = 18 - 9 = 9$, *As before*; And both these *Values* of a *equally true*, as to forming the given *Æquation*;

z. $36a - aa = 243$. For if $a = 9$, then $aa = 81$, and $= 324$; but $324 - 81 = 243$, therefore $a = 9$.

Again, if $a = 27$, then will $aa = 729$, and $36a = 972$: But $972 - 729 = 243$, *consequently* it may be, $a = 27$.

Now either of these Values of a may be found by *Division*, those were in the other Two *Cases*, one of them being *First* nd by the *Theorem*.

Thus, let $36a - aa - 243 = 0$ And $9 - a = 0$

Then $9 - a = 0$) $36a - aa - 243 = 0$ ($a - 27 = 0$

$9a - aa$

$27a - 0 - 243$

$27 \qquad 243$

$(0) \qquad (0)$

Hence, if $a - 27 = 0$ Then $a = 27$ As before.

Notwithstanding all *Quadratick Æquations* of this Third Form e *Two Affirmative Roots*, (as in this) yet but one of those ts will give a true *Answer* to the *Question*, and that is to be en according to the *Nature* and *Limits* of the *Question*, as l be shewed further on.

Scholium.

rom the Work of the Three last Examples, it may be observed; the Sum of both the Roots will always be Equal to the Co-ient of their respective Æquations, with a contrary Sign.

hus. In *Example* 1. $aa + 32a = 4644$

Here $a = 54$ } Add

And $a = -86$ }

$2a = -32$

Example 2. $aa - 7a = 948{,}75$

Here $a = 34{,}5$ } Add

And $a = -27{,}5$ }

$2a = +7$

the last *Example* $36a - aa = 243$

ch was *changed* into $aa - 36a = -243$

Here $a = 9$ } Add

And $a = 27$ }

$2a = 36$

Thus, $aa+32a-4644=0$ And $a-54=0$.
Then *Divide* the given *Æquation* by its *first Root*, and the *Quotient* will *shew* the *second Value* of a.

Thus, $a-54=0)\ aa+32a-4644=0\ (a+86=0$
$aa-54a$
$+86a-4644$
$86a-4644$
(0)

Hence the *Second Value* of a is $=-86$, Or $86=-a$ which *seems impossible*, *viz.* that an *Affirmative Quantity* should be *Equal* to a *Negative Quantity*; yet even by this *Second Value* of a, and the same *Co-efficient*, the true *(or first) Equation* may be formed.

Thus, Let	1	$a=-86$
1 @ 2	2	$aa=+7396$, viz. $-86\times-86=+7396$
1 × 32	3	$32a=-2752$
2 + 3	4	$aa+32a=4644$. As at first.

Example 2.

Suppose	1	$aa-7a=948,75$ Then *per Theorem* 2.
1 C □	2	$aa-7a+\frac{49}{4}=948,75+\frac{49}{4}=961$
2 ww 2	3	$a-\frac{7}{2}$ (Or 3,5) $=\sqrt{961}=31$
3 + 35	4	$a=31+3,5=34,5$

Again, for the Second Value of a,
Let $aa-7a-948,75=0$. And $a-34,5=0$
Then, $a-34,5=0)\ aa-7a-948,75=0\ (a+27,5=0$
Consequently this second *Value* is $a=-27,5$
which will *form* the *Original Æquation*, $aa-7a=948,75$ if it be ordered as the last was.

Example 3.

Suppose $36a-aa=243$ Then *per Theorem* 3.
$a=18-\sqrt{324-243}$ *viz.* half 36 *squared* is 324 &c.
That is, $a=18-\sqrt{81}$ but $\sqrt{81}=9$.
Therefore $a=18-9=9$. Now this *Third Form* is called an *Ambiguous Æquation*, because it hath *Two Affirmative* Values of the unknown *Quantity (a)*, both which may be found without such *Division* as was used before.

For in this Case, $a = 18 + \sqrt{81}$, viz. $a = 18 + 9 = 27$, Or, $a = 18 - 9 = 9$, *As before*; And both these *Values* of a are *equally true*, as to forming the given *Æquation*; *Viz.* $36a - aa = 243$. For if $a = 9$, then $aa = 81$, and $36a = 324$; but $324 - 81 = 243$, therefore $a = 9$.

Again, if $a = 27$, then will $aa = 729$, and $36a = 972$: But $972 - 729 = 243$, *consequently* it may be, $a = 27$.

Now either of these Values of a may be found by *Division*, as those were in the other Two *Cases*, one of them being *First* found by the *Theorem*.

Thus, let $36a - aa - 243 = 0$ And $9 - a = 0$

Then $9 - a = 0$) $36a - aa - 243 = 0$ ($a - 27 = 0$

$9a - aa$

$27a - 0 - 243$

$27 \qquad 243$

(0) (0)

Hence, if $a - 27 = 0$ Then $a = 27$ As before.

Notwithstanding all *Quadratick Æquations* of this Third Form have *Two Affirmative Roots*, (as in this) yet but one of those *Roots* will give a true *Answer* to the *Question*, and that is to be *chosen* according to the *Nature* and *Limits* of the *Question*, as shall be shewed further on.

Scholium.

From the Work of the Three last Examples, it may be observed; that the Sum of both the Roots will always be Equal to the Coefficient of their respective Æquations, with a contrary Sign.

Thus. In *Example* 1. $aa + 32a = 4644$

Here $a = 54$ } Add

And $a = -86$ }

$2a = -32$

In *Example* 2. $aa - 7a = 948{,}75$

Here $a = 34{,}5$ } Add

And $a = -27{,}5$ }

$2a = +7$

In the last *Example* $36a - aa = 243$

Which was *changed* into $aa - 36a = -243$

Here $a = 9$ } Add

And $a = 27$ }

$2a = 36$

Hence it's *evident*, that if either of the *Roots* be found, the other may be *eaſily* had without *Diviſion*.

If the *Contents* of this *Section* be well underſtood, it will be *eaſy* to give a *Numerical Solution* to any *Quadratick Æquation* that happens to ariſe in *Reſolving* of *Queſtions*, &c. And as for giving a *Geometrical Conſtruction* of them, I think it not *proper* in this *Place*; becauſe I here ſuppoſe the *Learner* wholly ignorant of the firſt *Principles* of *Geometry*, therefore I ſhall refer that Work to the next Part.

CHAP. IX.

Of Analyſis, *or the Method of Reſolving* Problems; *Exemplified by Variety of Numerical* Queſtions.

N. B. *Here I adviſe the Learner to make uſe always of the ſame Letters, to repreſent the ſame* Data *in all Queſtions.*

Viz. { If *a repreſent any Number* / And *e repreſent a leſs Number* } or other *Quantity*,

Then let {
- $a+e=s$ Their *Sum*.
- $a-e=d$ Their *Difference*.
- $ae=p$ Their *Product*.
- $\frac{a}{e}=q$ Their *Quotient*.
- $aa+ee=z$ The *Sum* of their *Squares*.
- $aa-ee=x$ The *Difference* of their *Squares*.

Any *Two* of theſe *ſix* (*s*, *d*, *p*, *q*, *z*, *x*) *being given*, thence find the *Reſt*; which admits of *Fifteen Variations*, or *Queſtions*.

Queſtion 1. Suppoſe *s* and *d* were given, and it were required by them to find *a*. *e p*. *q*. *z*. and *x*.

Let	1	$a+e=s$	and ſuppoſe $s=240$ Then
	2	$a-e=d$	$d=192$
1 + 2	3	$2a=s+d=432$	
3 ÷ 2	4	$a=\frac{s+d}{2}=216$	Here *a* is found
1 — 2	5	$2e=s-d=48$.	

5

$5 \div \frac{1}{2}$	6	$e = \frac{s-d}{2} = 24$ Here e is found.
4×6	7	$ae = \frac{ss-dd}{4} = p = 5184$ Here p is found.
$4 \div 6$	8	$\frac{a}{e} = \frac{s+d}{s-d} = q = 9$ Here q is found.
$4 \odot 2$	9	$aa = \frac{ss+2sd+dd}{4} = 46656$.
$6 \odot 2$	10	$ee = \frac{ss-2sd+dd}{4} = 576$
$9+10$	11	$aa+ee = \frac{ss+dd}{2} = z = 47232$ z found.
$9-10$	12	$aa-ee = sd = x = 46080$ x found.

Question 2. *Let* s and p be *given*; To find the *Reſt*.

That is,	1	$a+e = s = 240$
	2	$ae = p = 5184$ } Quere $a.e.d.q.z.x$.
$1 \odot 2$	3	$aa+2ae+ee = ss = 57600$
2×4	4	$4ae = 4p = 20736$
$3-4$	5	$aa-2ae+ee = ss-4p = 36864$
$5 \omega 2$	6	$a-e = \sqrt{ss-4p} = d = 192$
$1+6$	7	$2a = s+\sqrt{ss-4p}$
$7 \div \frac{1}{2}$	8	$a = \frac{s+\sqrt{ss-4p}}{2}$ Hence $a = 216$
$1-6$	9	$2e = s-\sqrt{ss-4p}$
$9 \div \frac{1}{2}$	10	$e = \frac{s-\sqrt{ss-4p}}{2}$ Hence $e = 24$.
$8 \div 10$	11	$\frac{a}{e} = \frac{s+\sqrt{ss-4p}}{s-\sqrt{ss-4p}} = q = 9$.
$8 \odot 2$	12	$aa = \frac{ss+s\sqrt{ss-4p}}{2} : -p$
$10 \odot 2$	13	$ee = \frac{ss-s\sqrt{ss-4p}}{2} : -p$
$12+13$	14	$aa+ee = ss-2p = z = 47232$
$12-13$	15	$aa-ee = s\sqrt{ss-4p} = x = 46080$

Question 3. Suppose s and q are *given*. To find the *Rest*.

Viz. { 1 $a+e=s.=240$; 2 $\frac{a}{e}=q=9$ } Quere $a.e.d.p.z.x.$

$2\times e$	3	$a=qe$
$1-3$	4	$e=s-qe$
$4+qe$	5	$qe+e=s$
$5\div q+1$	6	$e=\frac{s}{q+1}$ For $q+1\times e=qe+e$
$1-6$	7	$a=s-\frac{s}{q+1}=\frac{qs}{q+1}$
6×7	8	$ae=\frac{qss}{qq+2q+1}=p$
$7-6$	9	$a-e=\frac{qs-s}{q+1}=d$
$7\circledcirc 2$	10	$aa=\frac{qqss}{qq+2q+1}$
$6\circledcirc 2$	11	$ee=\frac{ss}{qq+2q+1}$
$10+11$	12	$aa+ee=\frac{qqss+ss}{qq+2q+1}=z$
$10-11$	13	$aa-ee=\frac{qqss-ss}{qq+2q+1}=x$

Question 4. Let s and z be *given*; To find the *Rest*.

Viz. { 1 $a+e=s.=240$; 2 $aa+ee=z.=47232$ } Quere $a.e.d.p.q.x.$

$1\circledcirc 2$	3	$aa+2ae+ee=ss$
$3-2$	4	$2ae=ss-z$
$2-4$	5	$aa-2ae+ee=2z-ss$
$5\ \text{ww}\ 2$	6	$a-e=\sqrt{2z-ss}=d$

1 + 6	7	$2a = s + \sqrt{2z - ss}$
7 ÷ 2	8	$a = \frac{s + \sqrt{2z - ss}}{2}$
1 − 6	9	$2e = s - \sqrt{2z - ss}$
9 ÷ 2	10	$e = \frac{s - \sqrt{2z - ss}}{2}$

The Rest are found just as in the 2d *Question*; the 8 and 10 *Steps* here, being the very same with the 8 and 10 *Steps* there.

Question 5. When s and x are *given*; To find the *Rest*.

Viz.	1	$a + e = s = 240$ } Quere $a.e.d.p.q.z.$
	2	$aa - ee = x = 46080$ }
2 ÷ 1	3	$a - e = \frac{x}{s} = d$ viz. $a+e)\ aa-ee\ (a-e$
1 + 3	4	$2a = s + \frac{x}{s} = \frac{ss+x}{s}$
4 ÷ 2	5	$a = \frac{ss+x}{2s}$
1 − 3	6	$2e = s - \frac{x}{s} = \frac{ss-x}{s}$
6 ÷ 2	7	$e = \frac{ss-x}{2s}$
5 × 7	8	$ae = \frac{ssss-xx}{4ss} = p$
5 ÷ 7	9	$\frac{a}{e} = \frac{ss+x}{ss-x} = q$
5 ⊙ 2	10	$aa = \frac{s^4 - 2ssx + xx}{4ss}$
7 ⊙ 2	11	$ee = \frac{s^4 - 2ssx + xx}{4ss}$
10 + 11	12	$aa + ee = \frac{s^4 + xx}{2ss} = z$

Question

Queſtion 6. Suppoſe d and p are *given*; To find the *Reſt*.

Viz. { 1 $a-e=d=192$; 2 $ae=p=5184$ } Quere $a.e.s.q.z.x.$

1 ⊙ 2	3	$aa-2ae+ee=dd$
2 × 4	4	$4ae=4p$
3 + 4	5	$aa+2ae+ee=dd+4p$
5 ww 2	6	$a+e=\sqrt{dd+4p}=s$
6 + 1	7	$2a=d+\sqrt{dd+4p}$
7 ÷ 2	8	$a=\frac{d+\sqrt{dd+4p}}{2}$
6 − 1	9	$2e=\sqrt{dd+4p}:-d$
9 ÷ 2	10	$e=\frac{\sqrt{dd+4p}:-d}{2}$
8 ÷ 10	11	$\frac{a}{e}=\frac{d+\sqrt{dd+4p}}{\sqrt{dd+4p}:-d}=q$
8 ⊙ 2	12	$aa=\frac{dd+2p+d\sqrt{dd+4p}}{2}$
10 ⊙ 2	13	$ee=\frac{dd+2p-d\sqrt{dd+4p}}{2}$
12 + 13	14	$aa+ee=dd+2p=z$
12 − 13	15	$aa-ee=d+\sqrt{dd+4p}=x$

Queſtion 7. Let d and q be *given*; To find the *Reſt*.

Viz. { 1 $a-e=d=192$; 2 $\frac{a}{e}=q=9$ } Quere $a.e.s.p.z.x.$

2 × e	3	$a=qe$
1 + e	4	$a=d+e$
3, 4	5	$qe=d+e$
5 − e	6	$qe-e=d$
6 ÷ $q-1$	7	$e=\frac{d}{q-1}$ For $q-1\times e=qe-e$
1 + 7	8	$a=d+\frac{d}{q-1}=\frac{qd}{q-1}$
7 + 8	9	$a+e=\frac{qd+d}{q-1}=s$

7 X

7×8	10	$ae = \frac{qdd}{qq-2q+1} = p$
8⊙2	11	$aa = \frac{qqdd}{qq-2q+1}$
7⊙2	12	$ee = \frac{dd}{qq-2q+1}$
11+12	13	$aa+ee = \frac{qqdd+dd}{qq-2q+1} = z$
11−12	14	$aa-ee = \frac{qqdd-dd}{qq-2q+1} = x$

Queſtion 8. *Suppoſe d* and *z given*; To find the *Reſt*.

Viz.	1	$a-e=d=192$ } Quere *a. e. s. p. q. x.*
	2	$aa+ee=z=47232$ }
1⊙2	3	$aa-2ae+ee=dd$
2−3	4	$2ae=z-dd$
2+4	5	$aa+2ae+ae=2z-dd$
5ww2	6	$a+e=\sqrt{2z-dd}=s$
1×6	7	$2a=d+\sqrt{2z-dd}$
7÷2	8	$a=\frac{d+\sqrt{2z-dd}}{2}$
6−1	9	$2e=\sqrt{2z-dd}:-d$
9÷2	10	$e=\frac{\sqrt{2z-dd}:-d}{2}$
8×10	11	$ae=\frac{z-dd}{2}=p$
8⊙2	12	$aa=\frac{z+d\sqrt{2z-dd}}{2}$
10⊙2	13	$ee=\frac{z-d\sqrt{2z-dd}}{2}$
12−13	14	$aa-ee=d\sqrt{2z-dd}=x$
8÷10	15	$\frac{a}{e}=\frac{d+\sqrt{2z-dd}}{\sqrt{2z-dd}:-d}=q$

Queſtion

Question 9. *Let d* and *x* be *given*; To find the *Rest*.

Viz.	1	$a-e=d=240$
	2	$aa-ee=x=46080$ } Quere *a*. *e*. *s*. *p*. *q*. *z*.
2 ÷ 1	3	$a+e=\frac{x}{d}=$ s. viz. $a-e)\ aa-ee\ (a+e$
1 + 3	4	$2a=\frac{dd+x}{d}$
4 ÷ 2	5	$a=\frac{dd+x}{2d}$
3 − 5	6	$e=\frac{x-dd}{2d}$
5 × 6	7	$ae=\frac{xx-d^4}{4dd}=p$
5 ÷ 6	8	$\frac{a}{e}=\frac{dd+x}{x-dd}=q$
5 ⊙ 2	9	$aa=\frac{d^4+2ddx+xx}{4dd}$
6 ⊙ 2	10	$ee=\frac{xx-2ddx+d^4}{4dd}$
9 + 10	11	$aa+ee=\frac{d^4+xx}{2dd}=z$

Question 10. *Let p* and *q* be *given*; To find the *Rest*.

Viz.	1	$ae=p=5184$
	2	$\frac{a}{e}=q=9$ } Quere *a*. *e*. *d*. *z*. *x*.
1 × 2	3	$aa=qp$ For $\frac{ae}{1}\times\frac{a}{e}=\frac{aae}{e}=aa$
3 ww 2	4	$a=\sqrt{qp}$
1 ÷ 2	5	$ee=\frac{p}{q}$ For $\frac{a}{1})\frac{ae}{1}(\frac{aee}{a}=ee$
5 ww 2	6	$e=\sqrt{\frac{p}{q}}$
4 + 6	7	$a+e=\sqrt{qp}+\sqrt{\frac{p}{q}}=s$

4—6

4—6	8	$a-e=\sqrt{qp}-\sqrt{\frac{p}{q}}=d$
3+5	9	$aa+ee=qp+\frac{p}{q}=z$
3—5	10	$aa-ee=qp-\frac{p}{q}=x$

Question 11. *Let* p and z be *given*; To find the *Rest*.

Viz.	1	$ae=p=5184$
	2	$aa+ee=z=47232$ } Quere $a.e.$ &c.
1 × 2	3	$2ae=2p$
2+3	4	$aa+2ae+ee=z+2p$
4ww 2	5	$a+e=\sqrt{z+2p}=s$
2—3	6	$aa-2ae+ee=z-2p$
6ww 2	7	$a-e=\sqrt{z+2p}=d$
5+7	8	$2a=\sqrt{z+2p}+\sqrt{z-2p}$
8 ÷ 2	9	$a=\frac{\sqrt{z+2p}+\sqrt{z-2p}}{2}$
5—7	10	$2e=\sqrt{z+2p}-\sqrt{z-2p}$
10 ÷ 2	11	$e=\frac{\sqrt{z+2p}-\sqrt{z-2p}}{2}$
9 ÷ 11	12	$\frac{a}{e}=\frac{\sqrt{z+2p}+\sqrt{z-2p}}{\sqrt{z+2p}-\sqrt{z-2p}}$
9⊙2	13	$aa=\frac{z+\sqrt{zz-4pp}}{2}$
11⊙2	14	$ee=\frac{z-\sqrt{zz-4pp}}{2}$
$aa-ee$	15	$aa-ee=\sqrt{zz-4pp}=x$

Question 12. *Let* p and x be *given*; To find the *Rest*.

Viz.	1	$ae=p=5184$
	2	$aa-ee=x=46080$ } Quere $a.e.$ &c.
1⊙2	3	$aaee=pp$

$2 \odot 2$	4	$aaaa - 2aaee + eeee = xx$
3×4	5	$4aaee = 4pp$
$4 + 5$	6	$aaaa + 2aaee + eeee = xx + 4pp$
$6 \omega 2$	7	$aa + ee = \sqrt{xx + 4pp} = z$
$2 + 7$	8	$2aa = x + \sqrt{xx + 4pp}$
$8 \div 2$	9	$aa = \frac{x + \sqrt{xx + 4pp}}{2}$
$9 \omega 2$	10	$a = \sqrt{\frac{x + \sqrt{xx + 4pp}}{2}}$
$7 - 2$	11	$2ee = \sqrt{xx + 4pp} : -x$
$11 \div 2$	12	$ee = \frac{\sqrt{xx + 4pp} : -x}{2}$
$12 \omega 2$	13	$e = \sqrt{\frac{\sqrt{xx + 4pp} : -x}{2}}$
$10 + 13$	14	$a + e = \sqrt{\frac{x + \sqrt{xx + 4pp}}{2}} + \sqrt{\frac{\sqrt{xx + 4pp} : -x}{2}} = s$
$10 - 13$	15	$a - e = \sqrt{\frac{x + \sqrt{xx + 4pp}}{2}} - \sqrt{\frac{\sqrt{xx + 4pp} : -x}{2}} = d$
$9 + 12$	16	$aa + ee = \sqrt{xx + 4pp} = z$

Question 13. Having q and z *given*; To find the *Rest*.

Viz.	1	$\frac{a}{e} = q = 9$	Quere $a . e .$ &c.
	2	$aa + ee = z = 47232$	
$1 \times e$	3	$a = qe$	
$3 \odot 2$	4	$aa = qqee$	
$2 - 4$	5	$ee = z - qqee$	
$4 + qqee$	6	$qqee + ee = z$	
$5 \div qq + 1$	7	$ee = \frac{z}{qq + 1}$	For $qq + 1 \times ee = qqee + ee$

2—7	8	$aa = z - \frac{x}{qq+1} = \frac{qqz}{qq+1}$
8 ww 2	9	$a = \sqrt{\frac{qqz}{qq+1}}$
7 ww 2	10	$e = \sqrt{\frac{z}{qq+1}}$
9+10	11	$a+e = \sqrt{\frac{qqz}{qq+1}} : + \sqrt{\frac{z}{qq+1}} = s$
9—10	12	$a-e = \sqrt{\frac{qqz}{qq+1}} : - \sqrt{\frac{z}{qq+1}} = d$
9×10	13	$ae = \sqrt{\frac{qqzz}{q^4+2qq+1}} = p$
8—7	14	$aa-ee = \frac{qqz-z}{qq+1} = x$

Question 14. When q and x are given; To find the *Rest*.

Viz.

	1	$\frac{a}{e} = q = 9$
	2	$aa-ee = x = 46080$

} Quere a. e. &c.

1×e	3	$a = qe$
3⊙2	4	$aa = qqee$
2+ee	5	$aa = x+ee$
4, 5	6	$qqee = x+ee$
6—ee	7	$qqee-ee = x$
7÷qq—1	8	$ee = \frac{x}{qq-1}$
2+8	9	$aa = x + \frac{x}{qq-1} = \frac{qqx}{qq-1}$
9 ww 2	10	$a = \sqrt{\frac{qqx}{qq-1}}$
8 ww 2	11	$e = \sqrt{\frac{x}{qq-1}}$
10+11	12	$a+e = \sqrt{\frac{qqx}{qq-1}} : + \sqrt{\frac{x}{qq-1}} = s$

10—11	13	$a-e=\sqrt{\frac{qqx}{qq-1}}:-\sqrt{\frac{x}{qq-1}}=d$
10 × 11	14	$ae=\sqrt{\frac{qqxx}{qqqq-2qq+1}}=p$
8+9	15	$aa+ee=\frac{qqx-x}{qq-1}=z$

Question 15. When z and x are *given*; To find the *Reſt*.

Viz. { 1 $aa+ee=z=47232$; 2 $aa-ee=x=46080$ } Quere a. e. &c.

1+2	3	$2aa=z+x$
3÷2	4	$aa=\frac{z+x}{2}$
1—2	5	$2ee=z-x$
5÷2	6	$ee=\frac{z-x}{2}$
4 ww 2	7	$a=\sqrt{\frac{z+x}{2}}$
6 ww 2	8	$e=\sqrt{\frac{z-x}{2}}$
7+8	9	$a+e=\sqrt{\frac{z+x}{2}}+\sqrt{\frac{z-x}{2}}=s$
7—8	10	$a-e=\sqrt{\frac{z+x}{2}}-\sqrt{\frac{z-x}{2}}=d$
7 × 8	11	$ae=\sqrt{\frac{zz-xx}{4}}=p$
7÷8	12	$\frac{a}{e}=\frac{\sqrt{z+x}}{\sqrt{z-x}}=q$

These *Fifteen Queſtions* are propoſed in *Dr. Pell's Algebra*, but he *purſues* only the *firſt Queſtion* throughout, and breaks *Off* in the other *Fourteen*, after the *Values* of what I call a and e are found. But I have *proceeded* in every *One* of them

o find the *Values* of all the *unknown Quantities*, becauſe they *afford* ſuch *Variety*, as being well *obſerved* by a *Learner*, will e found very uſeful in the *Solution* of moſt *Queſtions*.

Note, I have *choſe* to uſe the ſame *Numbers* for the *reſpective Value* of each *Quantity* throughout all the *Queſtions*, becauſe hey will be more *Satisfactory* in *proving* the Work than *various Numbers* would have been. Not but that many *Numbers* may e *taken* at *Pleaſure*, provided that the *Number repreſented* by , be *Greater* than that by *e*, &c. I have omitted the *Numeical Calculations purely* for the *Learner* to *practice* on.

Queſtion 16. There are Two Numbers, the Sum of their Squares is 2368; And the Greater of them is in Proportion to he Leſs, As 6 To 1. What are theſe Numbers?

Let a=the Greater Number, e=the Leſſer, and z=2368.

Then	1	$aa+ee=z$	} *By the* Queſtion.
And	2	$a:e::6:1$	
2 ∵	3	$1a=6e$	
3 ⊙ 2	4	$aa=36ee$	
1 − 4	5	$ee=z-36ee$	
5 + 36ee	6	$37ee=z$	
6 ÷ 37	7	$ee=\frac{z}{37}=64$	
7 ω 2	8	$e=\sqrt{\frac{z}{37}}=8$	
8 × 6	9	$6e=6\sqrt{\frac{z}{37}}=48$	
3, 9	10	$a=48$	

Proof
If $a=48$
And $e=8$
$aa=2304$
$ee=64$
$aa+ee=2368$
And $48:8::6:1$

Queſtion 17. There are Three Numbers in Continued Proportion, the Sum of the Extreams is 156, and the Mean is 72; What are the Two Extreams?

That is, Suppoſe $a.m.e$ in ∴ and $m=72$.

Then	1	$a+e=s=156$	} *By the* Queſtion.
	2	$a:m::m:e$	} Quere $a.e.$ &c.
2 ∵	3	$ae=mm$	
1 ⊙ 2	4	$aa+2ae+ee=ss$	
3 × 4	5	$4ae=4mm$	

4—5

4—5	6	$aa-2ae+ee=ss-4mm$
6 ww 2	7	$a-e=\sqrt{ss-4mm}$
1+7	8	$2a=s+\sqrt{ss-4mm}$
8 ÷ 2	9	$a=\frac{s+\sqrt{ss-4mm}}{2}=108$
1—9	10	$e=\frac{s-\sqrt{ss+4mm}}{2}=48$

Or $a=48$, $e=108$

Question 18. There are Three Numbers in ∺ their Sum 74, and the Sum of their Squares is 1924; What are the Numbers?

That is, a, e, y are in ∺

Then { 1 $a+e+y=s=74$; 2 $aa+ee+yy=z=1924$ } Quere a, e, y.
3 $a:e::e:y$

5 ∵	4	$ay=ee$
1—e	5	$a+y=s-e$
2—ee	6	$aa+yy=z-ee$
4 × 2	7	$2ay=2ee$
6+7	8	$aa+2ay+yy=z+ee$
5 ⊙ 2	9	$aa+2ay+yy=ss-2se+ee$
8 and 9	10	$z+ee=ss-2se+ee$
10±	11	$2se=ss-z$
11 ÷ 2s	12	$e=\frac{ss-z}{2s}=24$
5,	13	$a+y=s-e=50$
13 ⊙ 2	14	$aa+2ay+yy=2500$
4 × 4	15	$4ay=4ee=2304$
14—15	16	$aa-2ay+yy=196$
16 ww 2	17	$a-y=\sqrt{196}=14$
13+17	18	$2a=50+14=64$
18— 2	19	$a=32$
13—19	20	$y=50-32=18$

Or $a=18$, $y=32$

Note, In all *Questions* about *Continual Proportionals*, (either *Arithmetical* or *Geometrical*) where *Three Terms* are sought the *Mean* is *Easiest* found first (as above;) and if all the Terms be *Affirmative*, then 'tis Equal whether the *First*, or *Last Term* be the *Greatest*.

Quest

Queſtion 19. There are Three Numbers in ∴ their Sum is ; and if the Sum of the Extreams be Multiplied into the ean, that Product will be 1248; What are thoſe Numbers?

Viz.	1	$a : e :: e : y$
	2	$a+e+y=s=76$ } *by the* Queſtion.
	3	$ae+ye=p=1248$ }
1 ∴	4	$ay=ee$
1 × e	5	$ae+ee+ye=se$
5 − 2	6	$ee=se-p$
6 − se	7	$ee-se=-p$
7 $C\square$	8	$ee-se+\frac{1}{4}ss=\frac{1}{4}ss-p$
8 ω 2	9	$e-\frac{1}{2}s=\sqrt{\frac{1}{4}ss-p}$
9 + $\frac{1}{2}s$	10	$e=\frac{1}{2}s+\sqrt{\frac{1}{4}ss-p}=$ { 52. *Per Theorem* 3. 24. *Chap.* 8.
2 − 10	11	$a+y=52$
4 × 4	12	$4ay=4ee=2304$
11 ⊙ 2	13	$aa+2ay+yy=2704$
13 − 12	14	$aa-2ay+yy=400$
14 ω 2	15	$a-y=\sqrt{400}=20$
11 + 15	16	$2a=52+20=72$
16 ÷ 2	17	$a=36$ } { Or $a=16$
11 − 17	18	$y=52-36=16$ } { and $y=36$

N. B. If you take $e=\frac{1}{2}s+\sqrt{\frac{1}{4}ss-p}=52$ (at the 10th) Then it will be $76-52=24=a+y$, which is *poſſible*, *viz.* that the *Mean* ſhould be *Greater* than the *Sum* the *two Extreams*.

Therefore it muſt be $e=\frac{1}{2}s-\sqrt{\frac{1}{4}ss-p}=24$. (*See Page* 201.

Queſtion 20. There are Three Numbers in Arithmetical greſſion, the Firſt being Added to Twice the Second, and ree times the Third, their Sum will be 62; and the Sum of heir Squares is 275; What are thoſe Numbers?

uppoſe	1	a, e, y in *Arithmetical Progreſſion.*
And	2	$a+2e+3y=62$ } *by the* Queſtion.
	3	$aa+ee+yy=275$ }
Then	4	$a+y=2e$ *per Sect* 1. *Chap.* 6.
2 − 4	5	$2e+2y=62-2e$
5 ÷ 2	6	$e+y=31-e$
6 − e	7	$y=31-2e$
4 − 7	8	$a=4e-31$

8 ⊙ 2	9	$aa = 16ee - 248e + 961$
7 ⊙ 2	10	$yy = 961 - 124e + 4ee$
9 + 10	11	$aa + yy = 20ee - 372e + 1922$
3 — 11	12	$ee = 372e - 20ee - 1647$
12 + 20ee	13	$21ee = 372e - 1647$
13 — 372e	14	$21ee - 372e = -1647$
14 ÷ 21	15	$ee - \frac{124}{7}e = -\frac{549}{7}$
15 C□	16	$ee - \frac{124}{7}e + \frac{3844}{49} = \frac{3844}{49} - \frac{549}{7} = \frac{1}{49}$
16 ww 2	17	$e - \frac{62}{7} = \sqrt{\frac{1}{49}} = \frac{1}{7}$ The Mean
17 + $\frac{62}{7}$	18	$e = \frac{62}{7} + \frac{1}{7} = 9$ Or $8\frac{5}{7}$
18 × 4	19	$4e = 36$ Or $\frac{346}{7}$
8, 19	20	$a = 36 - 31 = 5$ Or $34\frac{6}{7} - 31 = 3\frac{6}{7}$
18 × 2	21	$2e = 18$ Or $17\frac{3}{7}$
7, 21	22	$y = 31 - 18 = 13$ Or $31 - 17\frac{3}{7} = 13\frac{4}{7}$

Question 21. There are Three Numbers in Arithmetic Progression; the Square of the First Term being Added to the Product of the other Two is 576; the Square of the Mean being Added to the Product of the Two Extreams, makes 612; and the Square of the last Term being Added to the Product of the first into the second, is 792: What are those Numbers?

Suppose	1	a, e, y In *Arith. Progres.* As before
Then	2	$aa + ye = 576$
	3	$ee + ya = 612$ *By the* Question.
	4	$yy + ae = 792$
1 ∴	5	$a + y = 2e$ *Per Sect.* 1. *Chap.* 6.
5 × e	6	$ae + ye = 2ee$
2 + 4	7	$aa + ye + yy + ae = 1368$
7 — 6	8	$aa + yy = 1368 - 2ee$
2 — ee	9	$ya = 612 - ee$
9 × 2	10	$2ya = 1224 - 2ee$
8 + 10	11	$aa + 2ya + yy = 2592 - 4ee$
5 ⊙ 2	12	$aa + 2ya + yy = 4ee$
11, 12	13	$4ee = 2592 - 4ee$
13 + 4ee	14	$8ee = 2592$
14 ÷ 8	15	$ee = 324$
15 ww 2	16	$e = \sqrt{324} = 18$ The Mean
8,	17	$aa + yy = 1368 - 2ee = 720$
10,	18	$2ya = 1224 - 2ee = 576$
17 — 18	19	$aa - 2ya + yy = 720 - 576 = 144$

1 ww 2	20	$a-y=\sqrt{144}=12$
5 + 20	21	$2a=2e+12=42$
21 ÷ 2	22	$a=24$ Or $\begin{cases} a=12 \\ y=24 \end{cases}$
5 — 22	23	$y=2e-24=12$

Question 22. 'Tis required to find Two such Numbers, that the Sum of their Squares may be $8226\frac{1}{2}$; and their Product being added to the Square of the Lesser, may be $6921\frac{1}{2}$.

Viz.	1	$aa+ee=8226\frac{1}{2}$ } Quere a and e
	2	$ae+ee=6921\frac{1}{2}$ }
1 — 2	3	$aa-ae=2305$
3 ±	4	$ae=aa-1305$
4 ÷ a	5	$ae=\frac{aa-1305}{a}$
5 ⊙ 2	6	$ee=\frac{a^4-2610aa+1703025}{aa}$
1 — aa	7	$ee=8226,5-aa$
6. 7	8	$\frac{a^4-2610aa+1703025}{aa}=8226,5-aa$
8 × aa	9	$a^4-2610aa+1703025=8226,5\ aa-a^4$
9 + a^4	10	$2a^4-2610aa+1703025=8226,5\ aa$
10 ±	11	$2a^4-10836,5aa=-1703025$
11 ÷ 2	12	$a^4-5418,25aa=-851512,5$ (765
12 C□	13	$a-5418,25aa+7339358,26562=6487845$
13 ww 2	14	$aa-2709,125=\sqrt{6487845,765625}-2547$
4+27&c.	15	$aa=2709,125+2547,125$ (125
Suppose	16	$aa=2709,125+2547,125=5256,25$
Then	17	$a=\sqrt{5256,25}=72,5$
And 5,	18	$e=\frac{aa+1305}{a}=\frac{5256,25-1305}{72,5}=54,5$
Or let	19	$aa=2709,125-2547,125=162$
10 ww 2	20	$a=\sqrt{162}=12,72$ &c.
Then	21	$e=\frac{162-1305}{12,72}$ Which is *impossible*.
Therefore		$a=72,5$ }
And		$e=54,5$ } As at the 17th and 18th Steps.

This *Question* may be perform'd with less Trouble, by *Substituting Letters* for the known *Numbers*.

Viz. $\begin{cases} aa+ee=z \\ ae+ee=p \end{cases}$ Then let $z-p=d=aa-ae$, &c.

Question 23. It is required to find Three such Numbers, that the Sum of the First and Second, being Multiplied with the Third, may be 37824; and the Sum of the Second and Third, Multiplied with the First, may be 59944; also, that the Sum of the First and Third, being Multiplied with the Second, may be 52456.

Let a, e, y represent the *Three Numbers*.

Then { 1 | $ay + ey = 37824 = b$
2 | $ea + ya = 59944 = c$
3 | $ae + ye = 52456 = d$ } Quere a, e, y.

1 + 2 + 3 | 4 | $2ae + 2ay + 2ye = b + c + d$
Let | 5 | $z = b + c + d$
4 ÷ 2 | 6 | $ae + ay + ye = \frac{1}{2}z = \frac{b+c+d}{2}$
6 — 3 | 7 | $ay = \frac{1}{2}z - d$
7 ÷ a | 8 | $y = \frac{z-2d}{2a}$
6 — 2 | 9 | $ye = \frac{1}{2}z - c$
6 — 1 | 10 | $ae = \frac{1}{2}z - b = \frac{z-2b}{2}$
10 ÷ a | 11 | $e = \frac{z-2b}{2a}$
8 × 11 | 12 | $e = \frac{z-2d}{2a} \times \frac{z-2b}{2a} = \frac{zz-2dz-2bz+4bd}{4aa}$
9, 12 | 13 | $\frac{z-2c}{2} = \frac{zz-2dz-2bz+4bd}{4aa}$
13 × 4aa | 14 | $2zaa - 4caa = zz - 2dz - 2bz + 4bd$
14 ÷ | 15 | $aa = \frac{zz-2dz-2bz+4bd}{2z-4c} = 55696$
15 ww 2 | 16 | $a = \sqrt{55696} = 236$
11 | 17 | $e = \frac{z-2b}{2a} = 158$
8, | 18 | $y = \frac{z-2d}{2a} = 96$

Question 24. 'Tis required to find Two such Numbers, that their Sum being subtracted from the Sum of their Squares, may Leave 14. And if their Product be Added to their Sum, it may make 14.

Let a and e be put for the *Numbers*, and let $y = a + e$

Then { 1 | $aa + ee - y = 14$
2 | $ae + y = 14$ } By the *Question*.

$1+y$	3	$aa+ee=14+y$
$2-y$	4	$ae=14-y$
4×2	5	$2ae=28-y$
$3+5$	6	$aa+2ae+ee=42-y$
$6\omega 2$	7	$a+e=\sqrt{42-y}$
But	8	$a+e=y$ By *Substitution* above.
7, 8,	9	$y=\sqrt{42-y}$
$9\odot 2$	10	$yy=42-y$
$10+y$	11	$yy+y=42$
$11C\square$	12	$yy+y+\frac{1}{4}=42+\frac{1}{4}=42,25$
$12\omega 2$	13	$y+\frac{1}{2}=\sqrt{42,25}=6,5$
$13-\frac{1}{2}$	14	$y=6,5-\frac{1}{2}=6$
Consequent	15	$a+e=6$ By *Restitution* from above.
3, 14	16	$aa+ee=14+6=20$
5, 15	17	$2ae=28-12=16$
16—17	18	$aa-2ae+ee=4$
$18\omega 2$	19	$a-e=\sqrt{4}=2$
$15+19$	20	$2a=8$
$23\div 2$	21	$a=4$ Proof
15—21	22	$e=6-4=2$

If $a=4$ And $e=2$
Then $aa+ee-a-e=14$
And $ae+a+e=14$
According to the *Question*.

Question 25. Three Men discoursing of their Money, saith the First, if 100 *l.* were Added to my Money, it would be as much as both your Money put together; said the second Man, if 100 *l.* were Added to my Money, I should have Twice as much as both you have; saith the Third Man, if 100 *l.* were Added to my Money, I should have then three times as much Money as both you have: How much Money had each Man?

Let *a* represent the *First Man's* Money, *e* the *Second*, and *y* the *Third*.

Then	1	$a+100=e+y$
	2	$e+100=2a+2y$ } by the *Question*.
	3	$y+100=3a+3e$
$1-a$	4	$e+y-a=100=s$
$2-e$	5	$2a+2y-e=100=s$ } Quere *a*, *e*, *y*.
$3-y$	6	$3a+3e-y=100=s$
4, 6	7	$e+y-a=3a+3e-y$
7+	8	$2y=4a+2e$
5—8	9	$2a-e=s-4a-2e$
$9+4a-2e$	10	$6a+e=s=100$
$4+6$	11	$2a+4e=2s=2000$

$10 \times \frac{-}{4}$	12	$24a+4e=4s=400$
$12-11$	13	$22a=2s=200$
$13 \div \frac{-}{22}$	14	$a=\frac{s}{11}=\frac{100}{11}=9\frac{1}{11}\ l.$
$10-6a$	15	$e=-6a=100-\frac{600}{11}=\frac{500}{11}=45\frac{1}{11}\ l.$
$1 \div \frac{-}{2}$	16	$y=2a+e=\frac{200}{11}+\frac{500}{11}=\frac{700}{11}=63\frac{7}{11}\ l.$

Answer. The $\left\{\begin{matrix}First\\Second\\Third\end{matrix}\right\}$ *Man* had $\left\{\begin{matrix}9\ l.\ 1\ s.\ 9\frac{9}{11}\ d.\\45\ l.\ 9\ s.\ 1\frac{1}{11}\ d.\\63\ l.\ 12\ s.\ 8\frac{1}{11}\ d.\end{matrix}\right.$

Question 26. Three Men have each such a Sum of Money that if the First and Second Mens Money be Added to half of what the Third Man hath; that Sum will be 92 *l.* And if the Second and Third Mens Money be Added to one third Part of the First Man's Money, that Sum will be 92 *l.* Lastly, if one fourth part of the second Man's Money be Added to the First and Third Mens Money, that Sum will also be 92 *l.* How much was each Man's Money?

Put a for the 1*st Man's* Money, e for the 2*d*, and y for the 3*d*.

Then $\left\{\begin{matrix}1 & a+e+\frac{1}{2}y=s\\2 & \frac{1}{3}a+e+y=s\\3 & \frac{1}{4}e+a+y=s\end{matrix}\right\}$ By the *Question*. And $s=92$

1, 2	4	$a+e+\frac{1}{2}y=\frac{1}{3}a+e+y$
$4-e$	5	$a+\frac{1}{2}y=\frac{1}{3}a+y$
$5 \times \frac{-}{2} \times \frac{-}{3}$	6	$6a+3y=2a+6y$
$6+$	7	$4a=3y$
$2 \times \frac{-}{3}$	8	$a+3e+3y=3s$
$8-7$	9	$a+3e=3s-4a$
$9-a$	10	$3e=3s-5a$
$10 \div \frac{-}{3}$	11	$e=\frac{3s-5a}{3}$
$3 \times \frac{-}{4}$	12	$e+4a+4y=4s=368$
$12-2$	13	$3\frac{3}{3}a+3y=3s=276$
13, 7	14	$3\frac{3}{3}a+4a=3s=276$
$14 \times \frac{-}{3}$	15	$11a+12a=9s=828$
$15 \div \frac{-}{23}$	16	$a=\frac{9s}{23}=\frac{828}{23}=36\ l.$ The 1*st Man's* Money.
11	17	$e=\frac{3s-5a}{3}=\frac{96}{3}=32\ l.$ The 2*d Man's* Money.
$7 \div \frac{-}{3}$	18	$y=\frac{4a}{3}=\frac{144}{3}=48\ l.$ The 3*d Man's* Money.

Question 27. Four Men walking abroad, found a Purse of Shillings only, out of which every one took a Number at an Adventure; afterwards by comparing their Numbers together they found; that if the First took 25 Shillings from the Second, it would make his Number equal with what the Second had then left. If the Second took 30 Shillings from the Third, his Money would then be Triple to what the Third had left. And if the Third took 40 Shillings from the Fourth, his Money would then be Double to what the Fourth had left. Lastly, the Fourth taking 50 Shillings from the First, he would then have three times as much as the First had Left, and 5 Shillings more.

'Tis required to tell how many *Shillings* each Man had. Put a for the First Sum, e the Second, y the Third, and u the Fourth.

Then { By the *Question.*

	1	$a+25=e-25$
	2	$e+30=3y-90$
	3	$y+40=2u-80$
	4	$u+50=3a-145$
1+25	5	$a+50=e$
2−30	6	$3y-120=e$
5, 6	7	$a+50=3y-120$
7+120	8	$a+170=3y$
8 ÷ 3	9	$y=\frac{a+170}{3}$
3−40	10	$y=2u-120$
9, 10	11	$2u-120=\frac{a+170}{3}$
11+120	12	$2u=\frac{a+170}{3}+120=\frac{a+530}{3}$
12 ÷ 2	13	$u=\frac{a+530}{6}$
4−50	14	$u=3a-195$
13, 14	15	$3a-195=\frac{a+530}{6}$
15 × 6	16	$18a-1170=a+530$
16+	17	$17a=1700$
17 ÷ 17	18	$a=100$ The 1st
by the 5	19	$e=150$ 2d
by the 9	20	$y=90$ 3d
by the 14	21	$u=105$ 4th

} *Man's Number of Shillings.*

Question

Question 28. Four Men have each a Sum of Money, which being put all together makes 250 Pounds. And if to the first Man's Money be added 8 Pounds, it will be just as much as the second Man's Money decreased by 8 Pounds, and as much as 2 times the third Man's Money, and but as much as one eighth part of the fourth Man's Money; How much had each Man?

Let a, e, y, u represent the *Four Mens Money*.

Then	1	$a+e+y+u=s$
	2	$a+b=e-b$
	3	$yb=\frac{u}{b}=a+b$
$2+b$	4	$a+2b=e$
$3\div b$	5	$y=\frac{a+b}{b}$ Because $yb=a+b$
$5\times b$	6	$u=ba+bb$. For $\frac{u}{b}=a+b$
$4+5+6$	7	$e+y+u=a+2b+\frac{a+b}{b}+ba+bb$
$1-a$	8	$e+y+u=s-a$
7, 8	9	$a+2b+\frac{a+b}{b}+ba+bb=s-a$
$9\times b$	10	$ba+2bb+a+b+bba+bbb=bs-ba$
10 $\pm$	11	$2ba+bba+a=bs-bbb-2bb-b$
11 $\div$	12	$a=\frac{bs-bbb-2bb-b}{bb+2b+1}=16,691358$, &c.
by the 4,	13	$e=a+2b=32,691358$, &c.
by the 5,	14	$y=\frac{a+b}{b}=3,086419$, &c.
by the 6,	15	$u=ba+bb=197,530864$, &c.

by the Question. Let $s=250$ and $b=8$. Or any other Number at Pleasure.

	l.	*s.*	*d.*
That is $a=$	16	13	9,9259
$e=$	32	13	9,9259
$y=$	3	1	8,7405
$u=$	197	10	7,4073

Consequently $a+e+y+u=249$. 19 . 11,9997 which should be just 250 *l.* the Sum proposed in the Question. Now what it wants of that Sum, proceeds from the Imperfection of the Decimal Parts being not Continued on to more Places which would have brought it nearer the Truth, tho' not perhaps exactly so. *Sect.* 5. *Chap.* 5. *Part* 1.

Questio

Question 29. Several Merchants enter into Partnership, every one put into the Stock 65 times as many Pounds as there were Partners; with that Stock they Traded, and gained as many Pounds per 100 *l.* as there were Partners. Now if 10 *l.* 10 *s.* be Added to, and Substracted from their Gain, the Product of that Sum and Difference will be 6491 *l.* 6 *s.* 3 *d.*

Quere, How many *Merchants* there were, &c.

Let	1	$a =$ *The Number of Merchants.*
$1 \times \overline{65}$	2	$65a =$ *Every one's Sum be put into* Stock.
$2 \times a$	3	$65aa =$ *The whole Stock.*
And	4	$100 : a :: 65aa : \frac{65aaa}{100}$ *by the* Question.
Viz.	5	$\frac{65aaa}{100} =$ *The whole* Gain.
$5 + \overline{10,5}$	6	$\frac{65aaa}{100} + 10,5$
$5 - \overline{10,5}$	7	$\frac{65aaa}{100} - 10,5$
6×7	8	$\frac{4225aaaaaa}{10000} - 110,25 = 6491,3125$ *by the* Quest.
$8 \times \overline{10000}$	9	$4225a^6 - 1102500 = 64913125$
$9 +$	10	$4225a^6 = 66015625$
$10 \div \overline{4225}$	11	$a^6 = \frac{66015625}{4225} = 15625$
11 *ww* 6	12	$a = \sqrt{15625} = 5$ *The* Number *of* Merchants.
$12 \times \overline{65}$	13	$65a = 325$. *The* Number *of* Pounds *each put in.*

Question 30. Three Merchants join Stocks together; the First Man's Stock was Less than the Second Man's by 13 *l.* the Second and Third Man's Stock was 175 *l.* in Trading they gain 48 *l.* more than their whole Stock was; the First Man's proportional Part of the Gain was 78 *l.* What was each Man's Stock and part of the Gain?

Let *a*, *e*, *y* represent each *Man's Stock.*

Then {	1	$a + e + y = s$ The whole *Stock.*
	2	$s + 48 =$ The whole *Gain.*
And {	3	$a + 13 = e$ } By the *Question.*
	4	$e + y = 175$ }
$4 + a$	5	$a + e + y = 175 + a$
1, 5	6	$s = 175 + a$

6, 2

6, 2	7	$s+48=223+a$
But	8	$175+a : 223+a :: a : 78$ *Per* Queſtion
8 ∴	9	$aa+223a=78a+13650$
9—78a	10	$aa+145a=13650$
10 C□	11	$aa+145a+5256,25=18906,25$
11 ω 2	12	$a+72,5=\sqrt{18906,25}=137,5$
12—72,5	13	$a=137,5-72,5=65$
3,	14	$e=a+13=78$
4 — 14	15	$y=97$
Then	16	$65 : 78 :: 78 : 93\,l.\,12\,s.=e$'s Gain.
Again	17	$65 : 78 :: 97 : 116\,l.\,8\,s.=y$'s Gain.
Proof	18	$116l.8s.+93l.12s.+78l.=288l.$ The *Gain.*
	19	$65+78+97=240.$ The whole *Stock.*
18—19	20	$288-240=48$ The *Gain* more than the *Stock.*

Queſtion 31. A Father at his Death left his Three Sons a his Money in this manner; to the Eldeſt he gave half of it wanting 44 Pounds; to the Second he gave one third of it, an 14 Pounds more; to the Younger he gave the Remainder, whic was Leſs than the Share of the Second Son, by 82 Pounds What was each Son's ſhare?

Let a, e, y be the *three Shares*, and z=the whole *Sum.*

Then { 1 $a+e+y=z$; 2 $a=\frac{1}{2}z-44$; 3 $e=\frac{1}{3}z+14$; 4 $y=\frac{1}{3}z+14-82$ } By the *Queſtion.*

2+3+4	5	$a+e+y=\frac{2z}{3}+\frac{z}{2}-98$
1, 5	6	$z=\frac{2z}{3}+\frac{z}{2}-98$
6 × 3	7	$3z=2z+\frac{3z}{2}-294$
7 × 2	8	$6z=4z+3z-588$
8+	9	$z=588.$ The whole Sum that was left.
2, 9	10	$a=\frac{588}{2}-44=250.$ The *Eldeſt Son's ſhare.*
3, 9	11	$e=\frac{588}{3}+14=210.$ The *Second Son's*, &c.
4, 9	12	$y=\frac{588}{3}+14-82=128.$ The *Youngeſt*, &c.

Queſtion 32. A Man playing at Hazard or Dice, won th Firſt Throw juſt ſo much Money as he had in his Pocket; th Secon

cond Throw he won the Square Root of what he then had, and e Shillings more: the Third Throw he won the Square of all then had: after which his whole Sum was 112*l.* 16*s. What oneys had he when he began to play?*

Suppose	1	a = His *First Sum.* Then
1 × ½	2	$2a$ = His *Sum* after the First Throw.
And	3	$5 + \sqrt{2a}$ = The *Winnings* at the 2*d Throw.*
2 + 3	4	$2a + 5 + \sqrt{2a}$ = The *Sum* after the 2*d Throw.*
4 ⊙ 2	5	$4aa + 22a + 25 + 4a\sqrt{2a} + 10\sqrt{2a}$ = The *Winnings* at the 3*d Throw*: and therefore
4 + 5	6	$4aa + 24a + 30 + 4a\sqrt{2a} + 11\sqrt{2a} = 2256$ *Sh.*

But to *avoid* these *Surd Quantities*, let us, instead of *supposing* a = the *First Sum*, make a *second Trial*, viz.

Let	1	$2aa$ = The *First Sum.*
1 × ½	2	$4aa$ = The *Sum* after the *First Throw.*
Then	3	$2a + 5$ = The *Sum* won at the 2*d Throw.*
2 + 3	4	$4aa + 2a + 5$ = His *Sum* after the 2*d Throw.*
4 ⊙ 2	5	$16a^4 + 16a^3 + 44aa + 20a + 25$ = The *Winnings* at the 3*d Throw*; and therefore
4 + 5	6	$16a^4 + 16a^3 + 48aa + 22a + 30 = 2256$ *Sh.*

Yet again, to avoid these *high Equations*, let us make a Third Supposition; thus,

Let	1	$\frac{aa}{2}$ = The First Sum.
1 × ½	2	aa = The Sum after the *First Throw.*
Then	3	$a + 5$ = The Winnings at the 2*d Throw.*
2 + 3	4	$aa + a + 5$ = The Sum after the 2*d Throw.*
Substi.	5	$e = aa + a + 5$.
5 ⊙ 2	6	ee = The Winnings at the 3*d Throw.* Then
5 + 6	7	$ee + e = 2256$ Shillings by the Question.
7 C□	8	$ee + e + 0{,}25 = 2256{,}25$
8 ww 2	9	$e + 0{,}5 = \sqrt{2256{,}25} = 47{,}5$
9 − 0,5	10	$e = 47$
5, 10	11	$aa + a + 5 = 47$
11 − 5	12	$aa + a = 42$
12 C□	13	$aa + a + 0{,}25 = 42{,}25$
1 ww ½	14	$a + 0{,}5 = \sqrt{42{,}25} = 6{,}5$
14 − 0,5	15	$a = 6$
15 ⊙ 2	16	$aa = 36$
16 ÷ ½	17	$\frac{aa}{2} = \frac{36}{2} = 18$ { The Shilling he had in his Pocket when he began to *play*

Note, In resolving of the last *Question*, I have made Thr different Suppositions for the Thing sought, purely as an Instanc to shew the young Learner how well he ought to consider t Nature of the *Question*, when he first states it, and make choi of representing the Thing sought, so as to avoid running it in Surds, if possible, *viz.* as in the first Supposition of $a =$ the fi Sum, &c. Not but that such Æquations may be solved, as sh be shew'd in the next Chapter. However, it is most like an A tist to perform Things of this Nature the neareſt and easiest w they can be done.

Question 33. *Suppose there were two equal Circles, whe Peripheries* (viz. *Circumferences*) *are Divided into* 44310 *equ Parts; and that those Circles were so placed upon one Axis, to move the contrary way to each other; and suppose one of the to move but one of these equal Parts the first Day, Two Parts t second Day, Three Parts the third Day, and so on in Arithmeti Progression*, viz. 1, 2, 3, 4, 5, &c. *and the other to move eve Day the Cubes of those Parts*, viz. 1. 8. 27. 64. 125, &c. *of t same Parts; How many Parts, and how many Days must ea Circle move, before the same Two Points meet that were tog ther when they began to move?*

In order to give a ready *Solution* to this *Question*, (or any oth in this kind) it will be convenient to *premise* this *Lemma*.

Lemma.

The *Sum* of any *Series of Cubes* whose *Roots* are in *Arithmeti Progression* (*the First Term, and common Difference being Uni or* 1) is equal to the Square of the Sum of all those *Roots*.

As in these

Terms in *Arith.* &c.	Their *Cubes*.
1	1
2	8
3	27
4	64
5	125
6	216, &c.

$21 \times 21 = 441$ *Sum* of their *Cubes*.

Let	1	$a =$ The *Sum* of all the *Parts* the 1*st Circle* mo
Then	2	$aa =$ The *Sum* of all the *Parts* the 2*d* moves.
Consequen.	3	$aa + a = 44310$ By the *Quest.* (*per L*
2 *C*❑	4	$aa + a + 0{,}25 = 44310{,}25$

4 w

4 ww 2	5	$a + 0,5 = \sqrt{44310,25} = 210,5$
5 — 0,5	6	$a = 210$ { The *Number* of *Parts* the *First Circle* must move.
6 @ 2	7	$aa = 44100$ { The *Number* of *Parts* the *Second Circle* moves.

Next to find the *Number* of *Days* they moved, There is given
e *First Term* = 1, the *common Difference* = 1. And the *Sum*
all the *Terms* = 210, thence to find the *Laſt Term*, which in
is Caſe is the ſame with the *Number* of all the *Terms*.
Let $a = 1$ the *Firſt Terms*, $e = 1$ the *common Difference*, and
= 210 the *Sum* of all the *Terms*, To find $y =$ the *Laſt Term*.
per Sect. 1. Chap. 6.
Then $yy + ey = 2s + aa - ae$ by the 16 *Step*, Page 186.
That is, $yy + y = 210 \times 2 = 420$, &c.
Hence $y = 20$ the *Number* of *Days required*.

I ſhall now proceed to give an *Example* or two of the *Method* ed in *Arguing* about Unlimited *Queſtions*, viz. ſuch *Queſtions* hich admit of various *Anſwers*, ſuch as thoſe in *Alligation Al-rnate*, promiſed in Page 117.
In order to ſhorten that Work, it will be *convenient* for the *arner* to know the two Signs of *Compariſon*, $>$ and $<$. The gn $>$ is of **Greater than**, As, $b > a$ *ſignifies* that b is *Grea-r than* a. The Sign $<$ is of **Leſſer than**. As $b < d$ *ſigni-s* that b is *Leſſer than* d, &c.

Example 1.

Queſtion 33. *A Tobacconiſt hath three ſorts of Tobacco*, viz. *e of* 2*s.* 8*d. the Pound, another of* 20*d. the Pound, and a third rt of* 16*d. the Pound; of theſe he would make a Mixture to ntain* 56 *Pound, that may be ſold for* 22*d. the Pound: How ch of each ſort may he take?*

Let $a =$ the *Quantity* of that worth 32 *Pence* the Pound, = that of 20 *Pence* the Pound, And $y =$ that of 16 *Pence* the und;

hen $a + e + y = 56$
nd $32a + 20e + 16y + 1232$ } *viz.* each *Quantity Multiplied* into its own *Price*, Equals their *Sum Multiplied* into the *Mean Price*.

This *Question* being thus ſtated, it appears by *Rule* [illegible] *Page* 176, that it is capable of *Innumerable Anſwers*; becauſe for any one of theſe *Three Letters*, *a. e. y.* there may be taken any *Number* at pleaſure, provided it be *Leſs* than 56. But although that may be truly done, yet there are ſeveral Ways of Arguing about theſe ſorts of *Queſtions*, which will *limit* or *bound* them to all their *proper* or *poſſible Anſwers* in whole *Numbers*. Thus,

Let	1	$a + e + y = 56$	} As above.
And	2	$32a + 20e + 16y = 1232$	
$1 - a$	3	$e + y = 56 - a$	
$2 - 32a$	4	$20e + 16y = 1232 - 32a$	
3×16	5	$16e + 16y = 896 - 16a$	
$4 - 5$	6	$4e = 336 - 16a$	
$6 \div 4$	7	$e = 84 - 4a$ Hence $a \nabla = 21$	
$3 - 7$	8	$y = 3a - 28$ Hence $a \triangle = 9\frac{1}{3}$	

From the *Two Laſt ſteps* it appears, that the *Quantity* ſignified by *a*, ought to be *Leſs* than 21, and *Greater* than $9\frac{1}{3}$; That is any *Number* betwixt $9\frac{1}{3}$ and 21, may be taken for the *Value* of *a*. *Conſequently* there may be *Eleven Anſwers* to this *Queſtion* in whole *Numbers*.

Suppoſe $a = 10$ Then $e = 84 - 40 = 44$ *per* 7th Step.
And $y = 30 + 2 = 28$ *per* 8th Step.
Again, if $a = 11$ Then $e = 84 - 44 = 40$ *per* 7th Step.
And $y = 33 - 28 = 5$ *per* 8th Step. And ſo on for the reſt, which will be as in the following *Table*.

a	*e*	*y*	*a*	*e*	*y*	*a*	*e*	*y*
10	44	2	14	28	14	18	12	26
11	40	5	15	24	17	19	8	29
12	36	8	16	20	20	20	4	32
13	32	11	17	16	23			

Thus it will be *eaſy* to find out and collect all the *Limited Anſwers* to any *Queſtion (of this kind)* wherein there are only *Three Quantities* propos'd to be *Mix'd*: But when there are More than *Three*, then the Work requires a little more Trouble; becauſe the *ſingle Limits* of all the *Quantities* above *Two* muſt be found. That is, if there are *Four Quantities* concern'd in the *Queſtion*, the *Limits* of *Two* of them muſt be found: If *Five Quantities* are concern'd, then the *Limits* of *Three* of them muſt be found, *&c.* As in the following *Queſtion*.

Queſtion 34. *Suppoſe it were requir'd to mix Four ſorts of ines together, viz. one ſort worth 7s. 4d. the Gallon; another worth 4 s. 7 d. the Gallon; a third ſort worth 3 s. 8 d. the allon; and a fourth ſort worth 2 s. 9 d. the Gallon; How ch of each ſort may be taken to make a Mixture of 36 Gallons, as that the whole Quantity may be ſold for 5 s. 6 d. the Gal-, without Loſs,* &c.

Firſt let all theſe ſeveral *Rates*, and the *Mean Rate*, be reduced one *Denomination, viz.* into *Pence*.

iz. $\left\{\begin{matrix} 7s.\ 4d. = 88\ d. & 4s.\ 7d. = 55\ d. \\ 3s.\ 8d. = 44d. & 2s.\ 9d. = 33\ d. \end{matrix}\right\}$ And $5s.\ 6d. = 66d.$

Then put $a =$ the *Quantity* of that worth 88 *d.* the *Gallon*; = that of 55 *d.* the *Gallon*; $y =$ that of 44 *d.* the *Gallon*; and = that of 33 *d.* the *Gallon.*

Then	1	$a + e + y + u = 63$ By the Queſtion.
And	2	$88a + 55e + 44y + 33u = 4158 = 93 \times 66$
$1 - a$	3	$e + y + u = 63 - a$
$2 - 88a$	4	$55e + 44y + 33u = 4158 - 88a$
3×33	5	$33e + 33y + 33u = 2079 - 33a$
$4 - 5$	6	$22e + 11y = 2079 - 55a$
$6 \div 11$	7	$2e + y = 189 - 5a$ Hence $a < \frac{189}{5} = 37\frac{4}{5}$
3×55	8	$55e + 55y + 55u = 3465 - 55a$
$8 - 4$	9	$11y + 22u = 33a - 693$
$9 \div 11$	10	$y + 2u = 3a - 63$ Hence $a > \frac{63}{3} = 21$

From the 7th and 10th *Steps* it appears, that the *Quantity* of at ſort of *Wine* denoted by *a*, muſt be *Leſs* than $37\frac{4}{5}$ *Gallons*, d *Greater* than 21 *Gallons:* That is, it may be $a =$ any *Num-* of *Gallons* betwixt 21 and $37\frac{4}{5}$.

Whence it follows, that there may be collected 16 *Anſwers* to is *Queſtion* from the *Limits* of *a* only.

Next to find the *Limits* of *e*, *y*, and *u*.

Suppoſe	11	$a = 22$ Then will $5a = 110$. And $3a = 66$
But	12	$2e + y = 189 - 5a = 79$ *per* 7th Step.
$12 - 2e$	13	$y = 79 - 2e$ Hence $e < \frac{79}{2} = 39\frac{1}{2}$
Again	14	$e + y + u = 63 - a = 41$ *per* 3d Step.
$14 - e$	15	$y + u = 41 - e$
$15 - 13$	16	$u = e - 38$ Hence $e > 38$

From

From the 13th and 16th Steps it appears, that if $a = 22$ The $e = 39$. $y = 79 - 2e = 1$. And $u = e - 28 = 1$

Again,

Suppose	17	$a = 23$ Then $5a = 115$. And $3a = 69$.
But	18	$2e + y = 189 - 5a = 74$. *per* 7th Step.
$18 - 2e$	19	$y = 74 - 2e$ Hence $e = < \frac{74}{2} = 37$.
Again	20	$e + y + u = 63 - e = 40$. *per* 3d Step.
$20 - e$	21	$y + u = 40 - e$
$21 - 19$	22	$u = e - 34$. Hence $e > 34$.

From the 19th and 22d Steps it appears, that if $a = 13$, th e may be either 35 or 36.

Once more for a further *Illustration*.

Let	23	$a = 24$. Then $5a = 120$. And $3a = 72$
But	24	$2e + y = 189 - 5a = 69$. *per* 7th Step.
$24 - 2e$	25	$y = 69 - 2e$. Hence $e < \frac{69}{2} = 34\frac{1}{2}$
Again	26	$e + y + u = 63 - a = 39$ *per* 3d Step.
$26 - e$	27	$y + u = 39 - e$
$27 - 25$	28	$u = e - 30$. Hence $e > 30$.

From hence it appears, that if $a = 24$, then e may be eith 31 . 32 . 33 . or 34. *viz.* it may be any *Number* betwixt 30 a $34\frac{1}{2}$ by the 25th and 28th Steps, from whence the *Values* of y a u may be easily found.

That is, if
$e = 31$. Then $y = 7$. And $u = 1$
$e = 32$. $y = 5$. $u = 2$
$e = 33$. $y = 3$. $u = 3$
$e = 34$. $y = 1$. $u = 4$

Proceeding on in this manner with all the other *single Valu* of a, there may be found above 120 *Answers* to this *Question whole Numbers*: And if you please to put $a =$ *Fractions*, the may be found an *Innumerable* Set of *Answers*; whereas the *Ru* of *Alligation* in *Vulgar Arithmetick* affords but only one *Answ* in *Fractions*, to wit, that of $a = 31\frac{1}{2}$. $e = 10\frac{1}{2}$. $y = 10\frac{1}{2}$. $u = 10$ As may be easily try'd *per* Rule *pag.* 115, *&c.*

These Two *Examples* being well understood (*especially if t Last be thorowly pursued*) may suffice to shew the *Method Limiting* the *Answers* to all sorts of *Questions* of this kind. I sh therefore conclude this *Chapter* of *Questions* with giving a So tion to the Ænigma (or Riddle) propos'd (*but not answer'd*)

Mr. *John Kersey*, in the Close of the *Appendix* to his *Arithmetick*, which affords several pretty *Questions*, the *Solution* whereof will discover a *certain Sentence* consisting of *Three Words*, which must be found by the help of *Figures* placed (*or supposed to be placed*) over the *Twenty-four Letters* of the *Alphabet*.

Thus $\begin{cases} 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot \&c. \text{ called } \textit{Indices}. \\ a \cdot b \cdot c \cdot d \cdot e \cdot f \cdot g \cdot \&c. \text{ to the last } \textit{Letter}. \end{cases}$

so that if the *Index* to that *Letter* be once found, the *Letter* to which it belongs is *Consequently* known.

*****

The Ænigma.

1. If the Difference between the *Indices* of the Second *Letter* of the *Second Word*, and the *Third Letter* of the *First Word*, be *Multiplied* into the Difference of their *Squares*, the Product will be 576. And if their *Sum* be *Multiplied* into the *Sum* of their *Squares*, that *Product* will be 2336, the *Index* of the said Third Letter being the Greatest.

Let	1	$a =$ the Greater *Index*, or that of the 3d Letter.
And	2	$e =$ the Lesser, or that of the 2d Letter.
Then $\begin{cases} \\ \\ \end{cases}$	3	$a - e \times aa - ee = 576$ } By the Question.
	4	$a + e \times aa + ee = 2336$ }
$3\times$	5	$aaa - aae - aee + eee = 576$
$4\times$	6	$aaa + aae + aee + eee = 2336$
$6 - 5$	7	$2aae + 2aee = 1760$
$6 + 7$	8	$aaa + 3aae + 3aee + eee = 4096$
8 ɯ 3	9	$a + e = \sqrt[3]{4096} = 16$
$\div a + e$	10	$aa + ee = \frac{2336}{a+e} = \frac{2336}{16} = 146$
9 ⊖ 2	11	$aa + 2ae + ee = 256$
11 − 10	12	$2ae = 110$
10 − 12	13	$aa - 2ae + ee = 36$
13 ɯ 2	14	$a - e = \sqrt{36} = 6$
9 + 14	15	$2a = 22$ } *From hence it appears, that the* 3d
15 ÷ 2	16	$a = 11$ } *Letter of the* 1*st Word is* l, *and*
9 − 16	17	$e = 5$ } *the* 2d *Letter of the* 2d *Word is* e.

Nore, *In order to set down the Letters (as they become found) in their proper places, it may be convenient to supply the vacant places with Stars.*

Thus $\begin{cases} \text{First Word.} \\ \text{**}l\text{**} \end{cases}$	Second Word.	Third Word.
	e*	*****

2. The

2. The *Indices* last found, are the Two *Extreams* of F *Numbers* in *Arithmetical Progression*, the Lesser *Mean* being *Index* of the First Letter of the Third Word; and the Gre *Mean* is the *Index* of the Fourth and last Letter of the First Wo

Viz. 5.7.9.11 are the Four Terms in *Arith. Progression.*

Whence it appears, that *G* (whose Index is 7) is the First L ter of the Third Word; and that *i* (whose Index is 9) is Fourth or Last Letter of the First Word; which being pla down, will stand thus, ** *li* *e***. *G***.

3. The Second Letter of the Third Word is the same with Third Letter of the First Word; and the Fifth Letter of Third Word is the same with the Last Letter of the First Wo

Whence the Letters will stand thus, ** *li*. *e***. *Gl***

4. The *Sum* of the *Squares* of the Indices of the First and cond Letters of the First Word is 520. And the *Product* of same Indices is seven Ninths of the *Square* of the Greater Ind which is the Index of the said First Letter.

Let $a =$ the Greater, and $e =$ the Lesser Index.

Then	1	$aa + ee = 520$	According to the *Data.*
And	2	$ae = \frac{7}{9}aa$	
$2 \div a$	3	$e = \frac{7}{9}a$	
$3 \odot 2$	4	$ee = \frac{49}{81}aa$	
$1 - 4$	5	$aa = 520 - \frac{49}{81}aa$	
5×81	6	$81aa = 42120 - 49aa$	
$6 + 49aa$	7	$130aa = 42120$	
$7 - 130$	8	$aa = \frac{212}{130} = 324$	
$8 \text{ ɯ } 1$	9	$a = \sqrt{324} = 18$	Its Letter is *s.*
3, 9	10	$e = \frac{7}{9}a = 14.$	Its Letter is *o.*

Hence the Letter will stand thus, *Soli.* *e***. *Gl.**

5. The *Difference* between the two Last Indices, is the I of the First Letter of the second Word, *viz.* $18 - 14 = 4$ b the Index of the Letter *D.*

Then the Letters will stand thus, *Soli. De* ***. *Gl.**

6.

6. The Third and laſt Letter of the Second Word, Alſo the Third Letter of the Third Word, are the ſame with the Second Letter of the firſt Word
Hence the Letters will ſtand thus, *Soli Deo Glo* $\ast^{i}\ast$.

7. The *Sum* of the *Indices* of the Fourth Letter of the Third Word, and the Sixth or Laſt Letter of the ſame Word, being *Added* to their *Product* is 35. And the Difference of their Squares is 288. The *Index* of the Laſt Letter being the Leaſt.

Put a = the Greater, and e = the Leſſer *Index*, as before.

Then	1	$ae + a + e = 35$
And	2	$aa - ee = 288$ } By the *Date*.
$1 - a$	3	$ae + e = 35 - a$
$\div a + 1$	4	$e = \frac{35 - a}{a \times 1}$ For $e \times a + 1 = ae + e$
4 & 2	5	$ee = \frac{1225 - 70a + aa}{aa + 2a + 1}$
$2 + 5$	6	$aa = 288 \times \frac{1225 - 70a + aa}{aa + 2a + 1}$
$6 \times aa$ &c.	7	$a^4 + 2a^3 \times aa = 288aa + 576a + 288$ $+ 1225 - 70a + aa$
$7 \pm$	8	$a^4 + 2a^3 - 288aa - 506a = 1513$

This Laſt Æquation being Reſolv'd according to the Method which ſhall be ſhewed in the next Chapter, it will be $a = 17$. *the Letter; and from the 4th Step* $e = \frac{35 - a}{a + 1} = 1$ *the Index of the Letter a.*

Then theſe Two Letters being placed according to the *Data* above, are all that are required by the *Enigma* to Compleat theſe Words.

Soli Deo Gloria.

CHAP. X.

The *Solution* of *Adfected Æquations* in Numbers.

Before we proceed to the *Solution* of *Adfected Æquations*, it may not be amiſs to ſhew the *Inveſtigation (or Invention)* of thoſe *Theorems* or *Rules* for *Extracting* the *Roots* of Simple *Powers*, made uſe of in Chapter 11. Part 1.

I ſhall here make Choice of the ſame Letters to repreſent the *Numbers* both given and ſought; as in my *Compendium* of *Algebra*.

Viz. Let { *G*, always denote the given *Reſolvend*. *r* = { any Number taken as near the true *Root* as may be, whether it be *Greater* or *Leſs*. *e* = { the unknown Part of the *Root* ſought by which *r* is to be either *Increaſed* or *Decreaſed*.

Then if *r* be any *Number Leſs* than the true *Root*, it will he *r* + *e* = the *Root* ſought.

But if *r* be taken *Greater* then the true *Root* it will then be *r* − *e* = the *Root* ſought.

And put *D* for the *Dividend* that is *produced* from *G*, after it is *Leſſen'd* and *Divided* by *r* &c. *(into the Co-efficients of Adfected Æquations)* according as the *Nature* of the *Root* requires.

Theſe Things being premiſed, we may proceed to *Raiſing* the *Theorems*.

Section 1.

I. For the *Square Root*, viz. *aa* = *G*. Quere *a*.

Let	1	$r + e = a$
1 @ 2	2	$rr + 2re + ee = aa = G$
2 − *rr*	3	$2re + ee = G - rr$ Call it *D*, viz. $D = G - r$
Then	4	$\frac{D}{2r +} = e$ This ſhews the 1ſt *Method* of *Extracting* the *Square Root*, *Sect.* 5. *Ch.* 11. *Part*
3 ÷ 2	5	$re + \frac{1}{2}ee = \frac{G - rr}{2} = D$

Which gives this Theorem $\frac{D}{r + \frac{1}{2}e} = e$

The *Arithmeticial Opperations* of both theſe *Theorems*, you have in the *Examples* of *Section* 2. *Page* 126; To which I refer

fer the *Learner*, ſuppoſing him by this *Time* to underſtand them without any more Words than what is there expreſt.

II. To *Extract* the Cube *Root*; viz. $aaa = G$. Quere a.

Let		
1 @ 3	1	$r + e = a$ Suppoſing r *Leſs* than the true *Root*.
2 − rrr	2	$rrr + 3rre + 3ree + eee = aaa = G$
	3	$3rre + 3ree + eee = G - rrr$
÷ $3r$	4	$re + ee + \frac{eee}{3r} = \frac{G - rrr}{3r} = D$

Let $\frac{eee}{3r}$ be *Rejected* or *Caſt off*, as being of ſmall *Value*:

Then it will be, $re + ee = D$, which gives this following

Theorem $\left\{ \frac{D}{r + e} = e \right.$

By this *Theorem* or *Rule*, the 1ſt and 2d *Examples* in *Caſe* 1. *Page* 132. are perform'd; the which being compared with this *Theorem* may be very eaſily underſtood.

Again, Suppoſe $aaa = G$, (*As before*) And let r be taken *Greater* than the *True Root*.

Then		
1 @ 3	1	$r - e = a$
+	2	$rrr - 3rre + 3ree = a^3 = G$ { eee being Rejected, as above.
−	3	$3rre - 3ree = rrr - G$
÷ $3r$	4	$re - ee = \frac{rrr - G}{3r} = D$

Which gives this Theorem $\left\{ \frac{D}{r - e} = e \right.$

By this *Theorem* the *Third Example* in *Caſe* 2. *Page* 133. is perform'd.

III. To *Extract* the Biquadrate *Root*; viz. $a^4 = G$ Quere a.

Let		
1 @ 4	1	$r - e = a$ Suppoſing r *Leſs* than *Juſt*.
2 − r^4	2	$r^4 + 4rrre + 6rree + a^4 = G$ { Rejecting all the Powers of e above ee.
	3	$4rrre + 6rree = G - r^4$
÷ $2rr$	4	$2re + 3ee = \frac{G - r^4}{2rr} = e$

Which gives this Theorem $\left\{ \frac{D}{2r + 3e} = e \right.$

By this *Theorem* the *Biquadrate Root* of any *Number* may be *Extracted*. But as I have already ſaid, Page 134 thoſe *Extractions* may be very well perform'd by *Two Extractions* of the *Square Root*. *Vide Example Page* 135.

IV. To *Extract* the Surſolid *Root*, viz. $a^5 = G$. *Quere a.*

If r be taken *Leſs* than *juſt*, then $r + e = a$. As before.

And $\left\{\frac{G - r^5}{5r^3} = D\right.$. Which gives this Theorem $\left\{\frac{D}{r - 2e} = e\right.$

By this *Theor.* the *Surſolid Root Examp.* 1. *Page* 136, is *Extracted*.

But if r be taken *Greater* than *juſt*; Then $r - e = a$.

And $\left\{\frac{r^5 - G}{5r^5} = D\right.$. Which gives this Theorem $\left\{\frac{D}{r - 2e} = e\right.$

By this Laſt *Theorem* the *Example* in *Page* 137 is perform'd.

I preſume it neeeleſs to purſue the Raiſing of thoſe Theorems for Extracting the Roots of Simple Powers, any further; becauſe the Method of doing it is General, how high ſoever they are; and therefore it may be eaſily underſtood by what is already done.

Section 2.

Notwithſtanding I have already ſhewed the *Solution* of *Quadratick Æquations*, Two ſeveral Ways, *viz.* by caſting *off* the Loweſt *Term*: And by *Compleating* the *Square*, vide *Section* 2 *Page* 195, &c. Yet it may not be amiſs to ſhew, how thoſe *Æquations* may be *Reſolved* into *Numbers* by this Univerſal Method of Continued *Series*; wherein, if the Firſt r be taken Equal to the Firſt true *Root*, or *Single Side* of the *Reſolvend*. And every *Single Value* of e (*as it becomes found*) be ſtill *Added* to it, for a new r, Then thoſe *Roots* may be *Extracted* without repeating a Second *Operation*, As before in the *Single Powers*.

Caſe 1. Let $aa + 2ba = G$. 'Tis required to find the *Value* of a.

Put		
	1	$r + e = a$
1 ⊙ 2	2	$rr + 2re + ee = aa$
1 × $2b$	3	$2br + 2be = 2ba$
2 + 3	4	$rr + 2br + 2re + 2be + ee = aa + 2ba =$
4 − rr &c.	5	$2re + 2bee + ee = G - rr - 2br$
5 ÷ 2	6	$re + be + \frac{1}{2}ee = \frac{1}{2}G - \frac{1}{2}rr - br = D$

Which gives this Theorem $\frac{D}{r + b + \frac{1}{2}e} = e$

Suppo

pose $b = 364$ And $G = 38692865$
$= 6000$ Then $rr = 36000000$ And $2br = 4368000$
$36000000 + 4368000 = 40368000 > 38692865 = G$
erefore the First $r < 6000$. Let $r = 5000$ Then

1 str $= 5000$	$19346432,5 = \frac{1}{2}G$
$b = 364$	$-1432000, = \frac{1}{2}rr + br$
$r + b = 5364$	$5026432,5 = D$ (800 $= e$
$+ \frac{1}{2}e = 400$	46112
Diviſor 5764)	41523 (60 $= e$
$r + b = 6164$	37164
$+ \frac{1}{2}e = 30$	4359 (7 $= e$
Diviſor 6194)	43592,5 867 $= e$
$r + b = 6224$	(0)
$+ \frac{1}{2}e = 3,5$	
Diviſor 6227,5	

irſt $r = 5000$
$+ e = 867$ $\Big\} = 5867 = a$ As was required.

e 2. If $aa - 2ba = G$ Then proceeding as above, there ll ariſe this Theorem $\left\{ \frac{D}{r - b + \frac{1}{2}e} = e. \right.$ &c.

d in *Caſe* 3. *viz.* $2ba - aa = G$ you will have s Theorem $\frac{D}{b - r + \frac{1}{2}e}$ &c. As above

think it needleſs to trouble the Reader with the Work of ſe Two *Theorems* in *Numbers*; becauſe if the laſt Example of ſe 1. be underſtood, the other will be *eaſy*. Not but that *Method* of *Compleating* the *Square* is very ready and eaſy, as may obſerve by the Work in ſeveral Queſtions of this Chapter.

Section 3.

In the *Solution* of all *Adfected Æquations*, that are above *(or gher than) Quadraticks*, it will be the beſt way to take $r =$ *next neareſt Root* of the *Æquation:* And then it will $r + e = a$ if r be *Leſs* than *juſt*; Or $r - e = a$ r be *Greater* than *juſt* (*as at the beginning of this Chap.*) And all the Powers of the unknown Part of the *Root* (*viz.* e) ove its *Square* (ee) are to be *Rejected or caſt Off*; As before

in

in Raiſing the *Theorems* for the *Simple Powers*. And therefo it is, that to ſupply the want of thoſe *Powers* (above *ee* in *Theorem*) the *Operation* muſt be repeated: As in the *Exam* of *Extracting* the *Cube Root*, Page 133. *viz.* when the *Figu* in the Root conſiſt of more than *Three* Places, (*vide* Page 1 and 141.)

Suppoſe $aaa + ba = G$. Quere a.

Let		
	1	$r + e = a$ *viz.* Let r be ſuppoſed *Leſs* th
1 & 3	2	$rrr + 3rre + 3ree = aaa$
$1 \times b$	3	$br + be = ba$
2 + 3	4	$rrr + br + 3rre + be + 3ree = a^3 + ba$
$4 \div 3r$	5	$\frac{1}{3}rr + \frac{1}{3}b + re + \frac{be}{3r} + ee = \frac{G}{3r}$
5 — &c.	6	$re + \frac{be}{3r} + ee = \frac{G}{3r} - \frac{1}{3}rr - \frac{1}{3}b = D$

Which gives this Theorem $\left\{ \frac{D}{r + \frac{b}{3r} - e} = e \right.$

But if r be taken *Greater* than *juſt*, Then it will be $re + \frac{be}{3r} - ee = \frac{1}{3}rr + \frac{1}{3}b - \frac{G}{3r} = D$ Which produces this Theorem $\left\{ \frac{D}{r + \frac{b}{3r} - e} = e \right.$

By either of theſe Two *Theorems* the Value of a may be ea found. Or rather otherwiſe as in the following Example.

Let $aaa + 24a = 587914$ Here $b = 24$

Suppoſe the Firſt $r = 90$ Then $r^3 = 729000 >$ 5879 without the 24×90 being added to it: Therefore $r <$ Again, Suppoſe $r = 80$ Then $r^3 = 512000$ And $24r = 19$ But $512000 + 1920 = 519320 < 587914$ Hence > 7 but nearer to it than 90. Therefore

it muſt be		
	1	$r + e = a$ Leſs than *juſt*
1 & 3	2	$rrr + 3rre + 3ree = aaa$
1×24	3	$24r + 24e = 24a$
2, in Numb.	4	$512000 + 1920e + 240ee = aaa$
3, in Numb.		$1920 + 24e = 24a$
4 + 5	5	$513920 + 1922 4e + 240ee = 587914$
$5 - 513920$	7	$19224e + 240ee = 73994$
$7 \div 240$	8	$80,1e + ee = 308,31 = D$
$8 \div$	9	$e = \frac{D}{80,1 + e}$

Operat

Operation 80,1) 308,31 = *D* (3,7, = *e*
$+e$ = ,3 249,3
1. *Diviſor* 83,1) 59,01 First $r = 80$,
$+e$ = ,7 58,66 $+e = 3,7$
2. *Diviſor* 83,8) ,335 $r + e = 83,7$
Or rather New $r = 83,7$ for a Second *Operation*, which ing *involved* and *tryed* (as above) will be found greater than : Therefore

muſt be	1	$r - e = a$
1 × 3	2	$rrr - 3rre + 3ree = aaa$
1 × 24	3	$24r - 24e = 24a$
n Numb.	4	$586376,253 - 21017,07e + 251,1ee = aaa$
n Numb.	5	$2008,8 - 24e = 24a$ (914
4 + 5	6	$588385,053 - 21041,07e + 251,1ee = 587$
6 +	7	$21041,07e - 251,1ee = 471,053$
÷ 251,1	8	$83,7955e - ee = 1,87595778 = D$
8 ÷	9	$e = \frac{D}{83,7955 - e}$

Operation 83,7955) 1,87595778 (,0223 = *e*
$-e$ = ,02 1,675510
1. *Diviſor* 83,7755) ,2001747
$-e$ = ,002 ,1675470
2. *Diviſor* 83,7735) ,03290078
$-e$ = ,0003 ,02513196
Diviſor 83,7732) ,00776882 —
Having once found *half* the Places of *Figures* for the Value , it will be needleſs to form *New Diviſors (as above)*; for Reſt of the *Figures* may be as truly found by plain *Diviſion* . Thus
e laſt Diviſor is 83,7732) ,007768820 (,0223 = *e* } Add
7539588 (,0000927 }
ſt $r = 83,7$ 2292320 ,0223927 = *e*
$-e$ = ,0223927 1675464
$-e = 83,6776073 = a$ 6168560
5864124 &c.

But if more Exactneſs be required, you may make the w $r = 83,6776073$ And *proceed* with it to a *Third Operation*;

ration; which will *afford* Twenty Seven Places of *Figures* f the *Value* of *a* That is, every *Operation* will produce *Tri* the *Places* of *Figures* to thoſe of the *Precedent r.* And t *Tripling* the *Places* of *Figures* in the *Root*, at every *Operati* holds good, and is to be obſerved in the *Solution* of all *Adfect Æquations (how high ſoever they are)* according to this Meth of Reſolving them. *See Page* 141.

Example 2. Suppoſe $aaa - ba = G$. Quere a.

If $r + e = a$ Then $re - \frac{\frac{1}{3}be}{r} + ee = \frac{\frac{1}{3}G}{r} + \frac{1}{3}b - \frac{1}{3}rr =$

Which gives this *Theorem* $\left\{ r - \frac{\frac{1}{3}b}{r} + e = \frac{D}{\quad} = e \right.$

But if $r - e = a$ Then $re + \frac{\frac{1}{3}be}{r} + ee = \frac{\frac{1}{3}G}{r} + \frac{1}{3}b - \frac{1}{3}rr =$

which gives this *Theorem* $\left\{ r - \frac{\frac{1}{3}b}{r} + e = \frac{D}{\quad} \right.$

Or you may proceed otherwiſe, as in the Laſt *Example.*

Let $aaa - 6438a = 104785688$ Here $b = 6438$
Suppoſe the Firſt $r = 500$. $rrr = 125000000$ and $br = 3219$
Then $125000000 - 3219000 = 121781000$
But $121781000 > 104785688$ Therefore $r < 500$
Again, Suppoſe $r = 400$ $rrr = 64000000$. and $br = 2575$
Then will $64000000 - 2575200 = 6142800$
But $61424800 < 104785688$ Hence $r > 400$

Conſequently r is betwixt 400 and 500. But 500 is the *neareſt*; Therefore, Let $r = 500$ being *Greater* than *juſt.*

Then	1	$r - e = a$
$1 \odot 2$	2	$rrr - 3rre + 3ree = aaa$
$1 \times b$	3	$br - be = ba$
2,in Numb.	4	$125000000 - 750000e + 1500ee = aaa$
3,in Numb.	5	$3219000 - 6438e = 6438a$ (5
$4 - 5$	6	$121781000 - 743562e + 1500ee = 10$
$6 +$	7	$743562e - 1500ee = 16995312$
$7 \div 1500$	8	$495e - ee = 11330 = D$
$8 \div$	9	$e = \frac{D}{495 - e}$

Opera

Operation 495) 11330 (23,8 = *e*
$-e = 20$ 950

Diviſor 475) 1830 First $r = 500$
$-e = 3$ 1416 $-e = 23{,}8$

Diviſor 472) 414,0 $r - e = 476{,}2 = a$
377,6

Let New $r = 476$ for a 2d *Operation*. Then $r^3 = 107850176$ and $br = 3064488$ But $107850176 - 3064488 = 104785688$ the ſame with the *Reſolvend*. Conſequently $a = 476$ *juſt*.

Example 3. Let $ba - aaa = G$ Quere *a*.

If $r + e = a$ Then $\frac{\frac{1}{3}be}{r} - re - ee = \frac{\frac{1}{3}G}{r} + \frac{1}{3}rr - \frac{1}{3}b = D$

which gives this *Theorem* $\left\{ \frac{D}{\frac{\frac{1}{3}b}{r} - re} = e \right.$

But if $r - e = a$ Then $re - \frac{\frac{1}{3}b}{r} - ee = \frac{\frac{1}{3}G}{r} + \frac{1}{3}rr - \frac{1}{3}b = D$

which gives this *Theorem* $\left\{ \frac{D}{r - \frac{\frac{1}{3}b}{r} - e} = e \right.$

Or otherwiſe as before in the *Two* Laſt *Examples*. Thus

Let $123456a - aaa = 12272861$. Here $b = 123456$: Suppoſe the Firſt $r = 200$ Then $rrr = 8000000$. and $br = 24691200$. then $24691200 - 8000000 = 16691200$ but $16691200 > 12272861$. therefore *r* is here *Leſs* than *juſt*, becauſe the higheſt Power is —, or *Negative*.

Again, Suppoſe $r = 300$ then $r^3 = 27000000$ and $br = 37036800$ Then $37036800 - 27000000 = 10036800 < 12272861$ Conſequently $r < 300$ and $r > 200$

Let $r = 300$. being the next *neareſt*, but more than *juſt*.

Then		
1 @ 3	1	$r - e = a$
1 x *b*	2	$rrr - 3rre + 3ree = aaa$
in Numb.	3	$br - be = ba$
in Numb.	4	$27000000 - 270000e = +900ee$
5 — 4	5	$37036800 - 123456e$
6 —	6	$10036800 + 146544e - 900ee = 12272861$
7 ÷ 900	7	$146544e - 900ee = 2236061$
1 ÷ &c.	8	$162e - ee = 2484 = D$
	9	$e = \frac{D}{162 - e}$

Operation. 162) 2484 (16,6 = *e*
$-e =$ 10 152

1. *Divisor* 152) 964 First $r = 300$
$-e =$ 6 876 $-e = 16{,}6$

2. *Divisor* 146) 88,0 $r - e = 283{,}4 = a$
87,6)

Or New $r = 283$ which being *Involved*, &c. will appear to be the true *Root*. That is, $a = 283$ *just*.

Note, These are usually called the *Three Forms* of *Cubic Æquations*; and in the *Solution* of the *Third* or *Last Form*, viz. $ba - aaa = G$, you may meet with some seeming *Difficulties* especially in making *Choice* of the First r, because this *Æquation* is an *Ambigious Æquation*, and hath Two *Affirmative Roots* viz. a *Greater* and *Lesser Root*. But having once found either of them, the other may be easily obtained by *Division* only As in the *Quadratick Æquations*. Vide *Chapter* 8.

As for instance, in the *Last Example*, $a = 283$
And $123456a - aaa = 12272861$. *Make* these *Two Æquations* $= 0$. To wit, Let $a - 283 = 0$.
And $-aaa + 123456a - 12272861 = 0$.

Then, $a - 283$) $-aaa + 123456a - 12272861$ ($-aa - 283a + 43367$
$-aaa + 283aa$
$-283aa + 123456a$ ($-283a + 80089$
$-283aa + 80089a$
$+43367a - 12272861$ ($+43367$
$+43367a - 12272861$
(0) (0)

Hence it appears that $-aa - 283a + 43367 = 0$. *Consequently* $aa + 283a = 43367$ this *Æquation* being *Solved*, $a = 110, 2722$ &c. which is the *Lesser Root* of the aforesaid *Æquation* $ba - aaa - G$ &c.

After this *Manner* all the *possible* and *impossible Roots* of any *Æquation* may be easily *discovered*, any one of its *Roots* being once found. I shall therefore omit *inserting* more *Examples* of that kind.

Suppose $aaa + baa + ca = G$. Quere a.

Let $b = 74, c = 8729$. and $G = 560783$
By Trial (*as before*) it will be found that the *next near* $1 = 40$ being something *Less* than *just*.

There

Therefore	1	$r + e = a$
× c	2	$cr + ce = ca$
@ 2 : × b	3	$brr + 2bre + bee = baa$
@ 3	4	$rrr + 3rre + 3ree = aaa$
in Numb.	5	$349160 + 8729e$
in Numb.	6	$118400 + 5920e + 74ee$
in Numb.	7	$64000 + 4800e + 120ee$
+ 6 + 7	8	$531560 + 19449e + 194ee = 560783$
− 531560	9	$19449e + 194ee - 29223$
÷ 194	10	$100,2e + ee = 153,06 = D$
÷	11	$e = \frac{D}{100,2 + e}$

Operation.	100,2	153,06	(1,5 = e
+ e =	1	101,2	First $r = 40$
			+ $e = 1,5$
1. Divisor	101,2)	51,86	
+ e =	,5	50,85	$r + e = 41,5 = 0$
2. Divisor	101,7	1,01	

Or New $r = 41,5$ for a *Second Operation*, which being duly involved, &c. will be found more than *juſt*.

Therefore	1	$r - e = a$
	2	$cr - ce = ca$
Then	3	$brr - 2bre + bee = baa$
	4	$rrr - 3rre + 3ree = aaa$

Theſe being turn'd into *Numbers*, &c. As above, they will be $20037,75e - 198,5ee = 390,375$ which being *Divided* by 198,5 the *Co-efficient* of ee will become $100,946e - ee = 1,966624$ &c. $= D$.

Operation.	100,946	1,966624	(,019 = e
− e =	,01	1,00936	
1. Divisor	100,936)	957264	
− e =	,009	908343	
2. Divisor	100,927)	* 489210	(,0004847
* Here I proceed by plain *Diviſion*, without forming New *Diviſors*		403708	,0194847 = e
		855020	
		807416	
Laſt $r = 41,5$		476040	
− e =	,0194847	403708	
$r - e = 41,4805153 = a$		72332 &c.	

Let the Laſt *Æquations* in the *Ænigma*, *Chap.* 9. be here *propoſed* for a *Solution*.

Viz. $aaaa + baaa - caa - da = G$

$b = 2$, $c = 288$. $d = 506$. and $G = 1513$ Quere a

By Tryals it will be found, that the *next* neareſt $r = 2$ being ſomething more than *juſt*.

Therefore	1	$r - e = a$
1 × d	2	$dr - de = da$
1 ⊙ 2 : × c	3	$crr - 2cre + cee = caa$
1 ⊙ 3 : × b	4	$brrr - 3brre + 3bree = baaa$
1 ⊙ 4	5	$r^4 - 4rrre + 6rree = aaaa$

Theſe being turned into *Numbers*, and thoſe duly *Collected* according as the *Signs* of the *Æquation* direct, they will become

$50680 - 22374e + 2232ee = 1513$. which being *Divided* by 2232 the *Co-efficient* of ee,
will be $10e - ee = 22 = D$

Then $\left\{ \frac{D}{10 - e} = e \right.$

Operation, 10) 22 (3 = e
$-e = 3$ 21
Diviſor 7) 1
Firſt $r = 20$
$- e = 3$
$r - e = 17 = a$ *juſt*. See the End of Chap. 9

By what hath been already done about the *Solution* of the few *Æquations* (*being carefully obſerved*,) I preſume the *Learner* will eaſily *Conceive* how to proceed in the *Solution* of all kinds of *Æquations*, be they never ſo *High*, or *Adfected*; therefore I ſhall not here propoſe many *various Examples*, but only take them as they fall in *Courſe* when I come to the next *Part*, wherein you will (*perhaps*) find ſuch *Æquations* with their *Solutions* as are not common.

CHA

CHAP. XI.

Of *Simple* Intereſt, Annuities or *Penſions*, &c.

Intereſt or the Uſe paid for the *Loan* of *Money*, is either *mple*; Or *Compound*.

Section 1. Of *Simple* Intereſt.

Simple Intereſt, is that which is paid for the *Loan* of any *Prin-pal* or *Sum* of *Money*, Lent out for ſome *Time*, at any *Rate per nt*: Agreed on between the *Borrower* and the *Lender*; which, cording to the late *Laws* of *England*, ought to be *Six Pounds* the Uſe of 100 *l.* for one Year, and *Twelve Pounds* for the ſe of 100 *l.* for Two Years. And ſo on for a *Greater*, or *ſſer Sum*, proportionable to the Time propoſed.

There are ſeveral Ways of *Computing* (or *Anſwering Queſtions* out) *Simple Intereſt*; as by the Single and Double *Rule* of *ree* (See *Page* 96, &c.) others make uſe of *Tables* Compoſed at eral *Rates per Cent.* As Sir *Samuel Moreland* in his Doctrine *Intereſt*, both Simple and Compound, is all perform'd by *bles*; wherein he hath detected ſeveral Material *Errors* com-tted by Doctor *Newton*, Mr *Kerſey* upon *Wingate*, and Mr. *vil*, &c. in the Buſineſs of Computing *Intereſt*, &c. by their *bles*, too tedious to be here repeated.

But I ſhall in this *Tract* take other *Methods*, and ſhew that Computations relating to *Simple Intereſt* are grounded upon *ithmetick Progreſſion*; and from thence *Raiſe* ſuch General *eorems*, as will ſute with all *Caſes*. In order to that

P = Any *Principal* or *Sum* put to *Intereſt*.
R = The *Ratio* of the *Rate*, *per Cent. per Annum*.
t = The *Time* of the *Principal* Continuance at *Intereſt*.
A = The *Amount* of the *Principal*, and its *Intereſt*.

Note, The *Ratio* of the *Rate*, is only the *Simple Intereſt* of 1 *l.* one *Year*, at any given *Rate*; and it's thus found.

. 100 : 6 :: 1 : 0,06 = the *Ratio* at 6 *per Cent. per An.*
100 : 7 :: 1 : 0,07 = the *Ratio* at 7 *per Cent*, &c.
ain 100 : 7,5 :: 1 : 0,075 = the *Ratio* at 7 and $\frac{1}{2}$ *per Cent.*

And if the given *Time* be whole *Years*; then t = the *mber* of whole *Years*: But if the *Time* given, be either pure *rts* of a *Year*, or *Parts* of a *Year* mix'd with *Years*; thoſe *rts* muſt be turn'd into *Decimals*; and then t = thoſe *imals*, &c. Now the Common *Parts* of a *Year* may be eaſily

easily turn'd or converted into *Decimal Parts*, if it be considered

That *One* { *Day is the* $\frac{1}{365}$ *Parts of a Year* = 0,00274 *fere*
Month is the $\frac{1}{12}$ *Part of a Year* = 0,0833333 &
Quarter is the $\frac{1}{4}$ *Part of a Year* = 0,25

These Things being premised, we may proceed to *Raising* the *Theorems*.

Let R = the *Interest* of 1 *l.* for *one Year*. As before.
Then $2R$ = the *Interest* of 1 *l.* for *two Years*.
And $3R$ = the *Interest* of 1 *l.* for *three Years*.
$4R$ = the *Interest* of 1 *l.* for *four Years*. And so on for any *Number* of *Years* proposed.

Hence it is plain, That the *Simple Interest* of one Pound is a *Series* of *Terms* in *Arithmetick Progression Increasing*; whose First *Term* and *Common Difference* is R. And the *Number* of all the *Terms* is t. Therefore the Last *Term* will always be
tR = the *Interest* of 1 *l.* for any given *Term* signified by t.

Then { *As one Pound: Is to the Interest of* 1 *l.* :: *So is a Principal or given Sum: To its Interest.*

That is, 1 *l.* : tR :: P : tRP = *the Interest of* P. Then the *Principle* being *Added* to its *Interest*, their *Sum* will be = the *Amount* required: Which gives this *General Theorem*.

Theorem $tRP + P = A$

From whence the *Three* following *Theorems* are easily deduced

Theorem 2 $\left\{ \frac{A}{tR+1} = P. \right.$ *Theorem* 3. $\left\{ \frac{A-P}{tP} = R. \right.$

Theorem 4. $\left\{ \frac{A-P}{RP} = t. \right.$

These *Four Theorems* Resolve all *Questions* about *Simple Interest*.

Question 1. *What will* 256 *l.* 10 *s. Amount to in* 3 *Years, one Quarter,* 2 *Months, and* 18 *Days, at* 6 *per Cent. per Annum?*
Here is given P = 256,5. R = 0,06. And t = 3,46599
For 3 Years = 3 Quere *A. per Theorem*
one Quarter = 0,25
2 Months = 0,16667 = 0,08333 × 2
18 Days = 0,04932 = 0,00274 × 18

Hence t = 3,46599 : × 0,06 = 0,2079594 = tR
Then 0,2079504 × 256,5 = 53,341586 = tRP
And 53,341586 + 256,5 = 309841586 = $tRP + P$ = A
That is, 309,841586 = 309 *l.* 16 *s.* 10 *d.* being the Amount required.

Que

Queſtion 2. *What Principal or Sum being put to Intereſt, ill Raiſe a Stock of* 309 *l.* 16 *s.* 10 *d.* in Three Years, one arter, Two Months and 18 Days; at 6 *per Cent. per An-m?*

Or the ſame Queſtion othewiſe ſtated thus.

hat is 309 *l.* 16 *s.* 10 *d. due* 3 *Years, one Quarter,* 2 *Months* d 18 *Days hence, worth in ready Money; Abating or Diſcount-g* 6 *per Cent,* &c.

Here is given $A = 309{,}841586$ $R = 0{,}06$ $t = 3{,}46599$ *found as before)* Thence to find P. *Per Theorem* 2.

First $3{,}46599 \times 0{,}06 = 0{,}2079594 = tR$

Then $tR + 1 = 1{,}2079594)\ 309{,}841586 = A\ (256{,}5 = P$

hat is, 256,5 = 256 *l.* 10 *s.* the *Anſwer* required.

ueſtion 3. *At what Rate or Intereſt, per Cent,* &c. *will* 256 *l.* 0 *s. Amount to* 309 *l.* 16 *s.* 10 *d. In* 3 *Yeas, one Quarter, Two Months and* 18 *Days.*

Here is given, $P = 256{,}5$, $A = 309{,}841586$ and $t = 3{,}46599$ o find R. *Per Theorem* 3.

First $309{,}841586 - 256{,}5 = 53{,}341586 = A - P$

Next $3{,}46599 \times 256{,}5 = 889{,}026435 = tR$

And $tR = 889{,}026435)\ 53{,}341586\ (00{,}06 =$ the Ratio.

Then 1 *l.* : 0,06 :: 100 : 6 = the *Rate* required.

ueſtion 4. *In what Time will* 256 *l.* 10 *s. Raiſe a Stock (or Amount to)* 309 *l.* 16 *s.* 10 *d. at* 6 *per Cent.* &c.

ere is given, $P = 256{,}5$ $A = 309{,}841586$ and $R = 0{,}06$ o find t *Per Theorem* 4.

irſt $309{,}841586 - 256{,}5 = 53{,}341586 = A - P$

nd $256{,}5 \times 0{,}06 = 15{,}39 = PR$

hen $15{,}39)\ 53{,}341586\ (3{,}46599 = t$

hat is $t = 3$ Years and ,46599 *Decimal Parts* of a *Year*; hich may be brought into *Common Parts* of a *Year,* thus

0,46599
0,25 = *one Quarter*

0,21599

And 0,08333) 0,21599 (2 *Months.*
,16666

0,02074) ,04933 . (18 *Days.*

ence $t = 3$ *Years, one Quarter,* 2 *Months, and* 18 *Days*; e *Anſwer* required.

It muſt needs be eaſy to Conceive, that what is here done 6 *per Cent.* may be done at any other Rate of Intereſt, by orming the Ratio, *viz.* R accordingly.

Scholium

Scholium.

Altho' it be according to the Laws and Cuſtom of *England* to compute Intereſt at the Proportion of *6 per Cent.* (as above) yet he that takes up Money at Intereſt for any Time Leſs than Even or Compleat Years, pays more Intereſt than ſeems reaſonably Due, according to the Rules of Art.

As for inſtance; If 100 *l.* be forborn at Intereſt one Whole Year, it amounts to 106 *l.* But (I ſay) if it be paid at the Half Year's End, it ſhould not amount to 103; as appears from the following Proportion.

Let a = the Amounts due at the Half Year's End; Then it will be $100 : a :: a : 106$ the Amount at the Year's End. *Ergo* $aa = 10600$ And $a = \sqrt{10600} = 102{,}9563 = 102\,l.\ 19\,s.\ 1\frac{1}{2}d.$ which is Leſs than 103 *l.* by $10d.\frac{1}{2}d.$ And if it be paid in Leſs than Half a Year's Time, the Error muſt needs be the Greater.

Section 2. Of Anuities or Penſions *in Arrears Computed at Simple Intereſt.*

Annuities or *Penſions*, &c. are ſaid to be in *Arrears*, when they are payable or Due, either *Yearly*, or *Half-yearly*, &c. and are Unpaid for any *Number* of Payments. Therefore the Buſineſs is to compute what all thoſe *Payments* will amount unto, allowing any *Rate* of *Simple Intereſt* for their Forbearance, from the *Time* each particular Payment became due: Now in order to that,

Put
- u = the *Annuity*, *Penſion*, or Yearly *Rent*, &c.
- t = the *Time* of its *Continuance*, or being Unpaid.
- R = the *Ratio*, or *Intereſt* of 1 *l.* for 1 *Year*, As before.
- A = the *Amount* of the *Annuity* and its *Intereſt*.

Then if u = the *Firſt Year's Rent*, due without *Intereſt*.

And Ru = the *Intereſt*, $2u$ = the *Rent*. } Due at the *End* of the *Second Year*.

And $2Ru$ = the *Intereſt*, $3u$ = the *Rent*. } Due at the *End* of the *Third Year*.

$3Ru$ = the *Intereſt*, $4u$ = the *Rent*. } Due at the *End* of the *Fourth Year*.

$4Ru$ = the *Intereſt*, $5u$ = the *Rent*. } Due at the *End* of the *Fifth Year*.

And ſo on for any *Number* of *Years*. Hence it's *Evident*, that $Ru + 2Ru + 3Ru + 4Ru + 5u = A$ the Sum of all the *Rents* and their *Intereſt*, being forborn 5 Years.

From

From whence it follows, that $Ru+2Ru+3Ru+4Ru=A-tu$.

Here $t=5$. *Divide* by u, Then $R+2R+3R+4R=\frac{A-tu}{u}$

Next to find the *Sum* of this *Progression* (*See Page* 185.) Thus

Let $R+2R+3R+4R$ &c. $=z$. Then $1+2+3+4$ &c. $=\frac{z}{R}$

Here the *Sum* of the *First* and *Last Terms* are $4+1=5=t$

And the *Numbers* of all the *Terms* is $4=t-1$. Therefore

$\frac{t-1}{2}\times t=$ the *Sum* of all the *Terms*. That is, $\frac{tt-t}{2}=\frac{z}{R}$

Hence $\frac{ttR-tR}{2}=z$. *Consequently* $\frac{ttR-tR}{2}=\frac{A-tu}{u}$

Now from this *Æquation* it will be *easy* to deduce the following *Theorems*.

Theorem 1. $\left\{\frac{ttRu-tRu+2tu}{2}=A.\right.$ Or $\left\{\frac{ttu-tu}{2}R:+tu=A\right.$

Theorem 2. $\left\{\frac{2A}{ttR-tR+2t}=u.\right.$ *Theorem* 3. $\left\{\frac{2A-2tu}{ttu-tu}=R:\right.$

Let $\frac{2}{R}-1=x$. Then $t=\sqrt{\frac{2A}{Ru}+\frac{xx}{4}}:-\frac{1}{2}x\Big\}$ *Theorem* 4.

Question 1. *If* 250*l. Yearly Rent* (*or Pension*, &c.) *be forborn or unpaid Seven Years*; *What will it amount to in that Time, at* 6 per Cent. *for each Payment as it becomes due?*

Here is given $u=250$. $t=7$. And $R=0{,}06$ To find A. *Per. Th.* 1. First $250\times 7=1750=tu$. $1750\times 7=12250=ttu$

Again $12250-1750=10500=ttu-tu$. And $\frac{10500}{2}\times 0{,}06=315$

Lastly $315+1750=2065=A$. *Viz.* 2065*l.* is the *Answ.* required.

But if the *Annuity*, *Rent* or *Pension*, is to be paid by *Quarterly* or Half Yearly *Payments*, &c.

Then $\frac{0{,}06}{2}=0{,}03=R$ for Half Yearly Payments.

And $\frac{0{,}06}{4}=0{,}015=R$ for Quarterly; Or $0{,}045=R$ for Three Quarterly Payments. Example of Half Yearly Payments.

Suppose 250*l.* per Annum, *to be paid by Half Yearly Payments, were in Arrears, or unpaid for Seven Years*; *What would it Amount to allowing* 6 per Cent. per Annum *for each Payment as it became Due.*

In this Example there is given $u=125=\frac{250}{2}$ $t=14$ the Number of Payments; And $R=0{,}03=\frac{0{,}06}{2}$ Thence to find A.

First $125\times 14=1750=tu$. $1750\times 14=24500=ttu$.

Again $24500-1750=22750=ttu-tu$. Then $\frac{22750}{2}=11379$

And 11375 × 0,03 = 341, 25 Lastly 341, 25 + 1750 = 2091, 25.
That is, $A = 2091l.$ 5s. the Answer required.

N.B. Hence it may be *Observed*, that Half Yearly Payments are more Advantageous than Yearly:
For $209l.$ 5s. > $2065l.$ by $26l.$ 5s. *Consequantly*, Quarterly Payments are more Advantageous than Half Yearly Payments.

Question 2. *What Yearly Rent, Pension, &c. being forborn or unpaid Seven Years, will raise a Stock of* $2065l.$ *allowing* 6 per Cent. per Annum *for each Payment as it becomes due?*

Here is given $A = 2065$. $t = 7$ And $R = 0,06$. To find u.
Per Theorem 2.
First $7 \times 0,06 = 0,42 = tR$, and $0,42 \times 7 = 2,94 = ttR$.
Then $ttR - tR = 2,52$.
Lastly $ttR - tR + 2t = 16,52)\ 4130 = 2A\ (250 = u$.
That is, $250l.$ *per Annum*, &c. will Raise $2065l.$ the *Stock* required.

Question 3. *In what time will* $250l.$ *Yearly Rent, Raise a Stock of* $2065l$ *Allowing* 6 per Cent. *&c. for the Forbearance of the Payments as they become Due?*
Here is given $u = 250$. $A = 2065$ = And $R = 0,06$ To find t.
Per Theorem 4. First

$\frac{2}{R^{\circ}, \cdot \frac{2}{6}} = 33,3333$ And $33,3333 - 1 = 32,3333 = x = \frac{2}{R} - 1$

Then 16,16666 &c. $= \frac{1}{2} x$, 261,3605 &c. $= \frac{1}{4} xx$.
Again $\frac{41,32}{15} = 275,333 = 2A - Ru$ And $275,3333 + 261,3605$
$= 536,6938 = \frac{2A}{Ru} + \frac{1}{4}xx$. Then $\sqrt{}\ 536,6938 = 23,1666$.
Lastly $23,1666 - 16,1666 = 7 = t$ the Time required.

Question 4. *If* $250l.$ *Yearly Rent, being forborn Seven Years will Amount to* $2065l.$ *allowing Simple Interest for every Payment as it becomes due, what must the Rate of the Interest be* per Cent. &c.

Here is given $u = 250$. $A = 2065$. And $t = 7$. To find R:
Per Theorem 3.

Thus $\begin{cases} ttu = 12250 \cdot 4130 = 1A \\ -tu - 1750 \cdot 3500 = 2tu \end{cases}$

$ttu - tu = 10500)\ 630 = 2A - 2tu\ (0,06 = R$
Then 1 : 0,06 :: 100 : 6 the Rate required.

Section

Section 3. *The Preſent Worth of Annuities or Penſions, &c. Computed at Simple Intereſt.*

The Buſineſs of *Purchaſing Annuities*, or taking of *Leaſes* &c. for any aſſigned *Time*, depends upon the True *Æquating* of the *Principal* or Money laid out on the *Purchaſe*, with the *Annuity* or Yearly *Rent*, by allowing *(or Diſcompting)* the ſame *Rate* of *Intereſt* to both *Parties*. Which may be eaſily perform'd by duly applying the reſpective *Theorems* of the Two Laſt *Sections* together. As will fully appear by the following *Queſtion*.

Queſtion 1. *What is* 75*l. Yearly Rent, to continue Nine Years worth in ready Money*, at 6 per Cent. per Annum *Simple Intereſt?*

1. *Per Theorem* 1. of the *Laſt Section*, find what the propoſed Yearly Rent woul'd Amount to, if it were forborn 9 Years at 6 *per Cent.*

Thus $u = 75$. $t = 9$. And $R = 0{,}06$ Quere A.

$ttu = 6075$ Then 2) $5400\ (2700$ } Multiply

$tu = 675$ $R = 0{,}06$ }

$ttu - tu = 5400$ $\left.\begin{matrix} 162, \\ +tu = 675, \end{matrix}\right\} = 837 = A$

2. Then by *Theorem* 2. *Section* 1. find what Principal, being put to Intereſt for the ſame Time, and at the ſame Rate, will Amount to 837*l.* $= A$.

Thus $tR = 0{,}54 = 9 \times 0{,}06$. $tR + 1 = 1{,}54)837(543{,}5064 = P$.

That is, $P = 543\frac{1}{2}l.$ 10*s.* $1\frac{1}{4}d.$ Which is the Worth of 75*l.* a Year, as was required.

From the Work of theſe *Two Operations*, (duly Conſider'd) it muſt needs be *Eaſy* to Conceive, how the *Two Theorems* by which they were perform'd, may be *Combined* in One.

For 1. $\frac{ttRu - tRu + 2tu}{2} = A$. And 2. $PtR + P = A$.

Conſequently $PtR + P = \frac{ttRu - tRu + 2tu}{2}$ And from this *Æquation* may be deduced the following *Theorems*.

Theorem 1. $\left\{\frac{ttRu - tRu + 2tu}{2tR + 2} = P.\right.$ Or $\frac{ttR - tR + 2t}{2tR + 2} \times u = P$

By this *Theorem* all *Queſtions* of the ſame kind with the Laſt (viz. *that above*) may be eaſily and readily *Anſwered* at one *Operation*.

Theorem 2. $\left\{ \frac{2PtR+2P}{ttR-tR+2t} = u. \text{ Or } \frac{tR+1}{ttR-tR+2t} : \times 2P = u \right.$

Theorem 3. $\left\{ \frac{2P-2tu}{ttu-tu-2Pt} = R \right.$

Let $\frac{2}{R} - \frac{2P}{u} - 1 = x$. Then will $tt \pm xt = \frac{2P}{Ru}$

Which gives this *Theorem* 4. $\left\{ \sqrt{\frac{2P}{Ru} + \frac{xx}{4}} \pm \frac{x}{2} = t \right.$

By the *Second* and *Fourth Theorems*, Two very *Useful Questions* may be eaſily *anſwer'd*.

1. *As for Inſtance: If it be required to find what Annuity, or Yearly Rent*, &c. *may be purchaſed, for any propoſed Sum, to continue any aſſigned Time, allowing any Rate of Intereſt.*

This Queſtion may be Anſwered by Theorem 2.

2. *Again: If it be required to find how long any Yearly Rent, Penſion, or Annuity*, &c. *may be purchaſed (or enjoy'd) for any propoſed Sum, at any given Rate of Intereſt.*

All Queſtions of this kind are eaſily Anſwered *per* Theorem 4.

In theſe Queſtions it is ſuppoſed, that the Purchaſer or Yearly Rent, is to Commence or be immediately enter'd upon. But if it be required to find the Value or Purchaſe of an Annuity or Yearly Rent, *&c.* in Reverſion; That is, when it is not to be Enter'd upon until after ſome Time, or *Number* of *Years* are paſt. Then you muſt firſt find what the *Sum* propos'd to be laid out in the Purchaſe, would *Amount* to, if it were put to Intereſt during the time the Annuity, *&c.* is not to be in preſent Poſſeſſion; and make that Amount the Sum for the Purchaſe proceeding with it as in either of the two laſt Queſtions, *&c.*

Note, *From the Firſt Queſtion of this Section it will be eaſy to Conceive how to perform the Æquation of Payments, between Debtor or Creditor, at any Rate of Intereſt, without doing any Damage to either Party.*

That is, when ſeveral Sums of Money are to be paid, at ſeveral different Times, to find the Time when all the Payments may be truly diſcharg'd at once: As if one Sum were to be paid at the end of Two Months, another at Six Months, and perhaps a Third Sum at Eight Months End, *&c.* And if it were requir'd to find the Time when all thoſe Sums may be truly diſcharged at one Payment without Loſs, *&c.*

CHAP.

CHAP. XII.

Of Compound Intereſt, and Annuities, &c.

Compound Intereſt is that which ariſes from any *Principal* and *Intereſt* put together, as the *Intereſt* ſtill becomes Due; ſo t at every Payment, or at the *Time* when the Payments became e, there is Created a *New Principal*; And for that *Reaſon* it's lled *Intereſt* upon *Intereſt*, or *Compound Intereſt*.

As for Inſtance; Suppoſe 100*l.* were Lent out for Two ars, at 6 *per Cent*, *per Annum*, Compound Intereſt: Then, the End of the Firſt Year, it will only Amount to 106*l.* As Simple Intereſt. But for the Second Year this 106*l.* becomes ncipal, which wiz Amount to 112*l.* 7*s.* 2½*d.* at the Se- d Year's End, whereas by Simple Intereſt it would have A- unted to but 112*l.*

And altho' it be not lawful to Let out Money at Compound ereſt; yet in Purchaſing of Annuities or Penſions, &c. And ing Leaſes in Reverſion, it is very uſual to allow Compound ereſt to the Purchaſer for his ready Money; and therefore it ery requiſite to underſtand it.

Section 1. *Of* Compound *Intereſt.*

P = *the Principal put to Intereſt.*
t = *the Time of its Continuance.*
A = *the Amount of the Principal and Intereſt.*
} As before.

R = { *the Amount of* 1 *l. and its Intereſt for* 1 *Year, at any given Rate, which may be thus found.*

100 : 106 :: 1 : 1,06 = the *Amount* of 1 *l.* at 6 per *Cent.*
100 : 105 :: 1 : 1,05 = the *Amount* of 1 *l.* at 5 per *Cent.*
ſo on for any other aſſigned Rate of Intereſt.

en if R = the *Amount* of 1 *l.* for *One Year*, at any Rate.
RR = the *Amount* of 1 *l.* for *Two Years.*
RRR = the *Amount* of 1 *l.* for *Three Years.*
R^4 = the *Amount* of 1 *l.* for *Four Years.*
R^5 = the *Amount* of 1 *l.* for *Five Years.* Here $t = 5$

1 : R :: R : RR :: RR : RRR :: RRR : R^4 :: R^5 : R^4 : &c. in ∺.

t is { As one Pound : Is to the Amount of one Pound at one Year's End :: So is that Amount : To the Amount of one Pound at Two Year's End, &c.

Whence

Whence it is plain, that Compound Intereſt is grounded on a Series of Terms, Increaſing in Geometrical Proport Continued; wherein t (*viz.* the Number of Years) does ways aſſign the *Index* of the laſt and higheſt Term. *Viz.* the Power of R, which is Rt.

Again, As $1 : Rt :: P : PRt = A$ the Amount of P the Time, thar $Rt =$ the Amount of $1l$.

That is { *As one Pound : Is to the Amount of one Pound for given Time : : So is any propoſed Principal (or Sum To its Amount for the ſame Time.*

From the Premiſes, (*I preſume*) the Reaſon of the follo ing *Theorems*, may be very eaſily underſtood.

Theorem 1. $PRt = A$ As above.

From hence the Two following *Theorems* are eaſily deduc

Theorem 2. $\left\{\frac{A}{Rt} = P\right.$. *Theorem* 3. $\left\{\frac{A}{P} = Rt.\right.$

By theſe Three *Theorems*, all Queſtions about Compo Intereſt, may be truly Reſolved by the Pen only, *viz.* with Tables; Tho' not ſo readily as by the Help of Tables, Calc ted on Purpoſe, As will appear farther on.

Queſtion 1. *What will* 256*l*. 10*s*. *Amount to in Se Years, at* 6 per Cent. per Annum. *Compound Intereſt.*

Here is given $P = 256{,}5$. $t = 7$. and $R = 1{,}06$ wh being involved until its Index $= t$ (*viz.* 7.) will bec $R^7 = 1{,}50363$
Then $1{,}50363 \times 256{,}5 = 385{,}6811 = A = 385l.$ 13*s*. 7 which is the Anſwer required.

Queſtion 2. *What Principal or Sum of Money muſt be (or Let) out to Raiſe a Stock of* 385*l*. 13*s*. 7½*d*. *in Seven Ye at* 6 per Cent. per Annum *Compound Intereſt.*

Here is given $A = 385{,}6811$ $R = 1{,}06$ and $t = 7$ To find by *Theorem* 2.

Thus $Rt = 1{,}50363)\ 385{,}6811 = A\ (256{,}5 = P.$
That is, $P = 256l.$ 10*s*. which is the Principal or Sum was required.

Que

Queſtion 3. *In what Time will* 256*l*. 10*s*. *Raiſe a Stock (or Amount to)* 385*l*. 13*s*. 7½*d*. *allowing* 6 per Cent. per num *Compound Intereſt?*

e is given $P = 256,5$ $A = 385,68111$ $R = 1,06$ To find t

he Third *Theorem* $\left\{ Rt = \frac{A}{P} \frac{385,68111}{256,5} = 1,50363, \right.$

hich being continually Divided by $R = 1,06$ untill nothing ain, the Number of thoſe Diviſions will be $= 7 = t$

s 1,06) 1,50363 (1,41852 And 1,06) 1,041852 (1,338225 in 1,06) 1,338225 (1,262477. And ſo on untill it be- e 1,06) 1,06 (1. which will be at the Seventh Diviſion. refore it will be $t = 7$ the Number of Years required by the ſtion.

Queſtion 4. *If* 255*l*. 10*s*. *will Amount to (or Raiſe a e of)* 385*l*. 13*s*. 7½*d*. *in Seven Years Time; What muſt Rate of Intereſt be,* per Cent. per Annum.

e is given $P = 256,5$, $A = 385,68111$ and $t = 7$ Quere R.

heorum 3. $\left\{ \frac{A}{P} = Rt = 1,50363. \right.$ As before in the Laſt ſtion. And if $Rt = R^7 = 1,50363$ Then $R = \sqrt[7]{1,50363}$ h may be thus *Extracted.*

Put	1	$r + e = R$ Then
⊙ 7	2	$r^7 + 7r^6e + 21r^5ee = R^7 = 1,50363 = G$
$- r^7$	3	$7r^6e + 21r^5ee\ G - r^7$
$\div 7r^5$	4	$re + 3ee = \frac{G - r^7}{7r^5} = D$
	5	$e = \frac{D}{r + 3e}$ Let $r = 1$ Then $D = 0,0719$

ation $r = 1,00)\ 0,0719\ (0,06 = e$
$+ 3e = \ ,18\ \ 708$
Diviſor 1,18 (11) to be rejected.

First $r = 1,00$
$+ e = 0,06$ $\Big\} = 1,06 = R$

n 1 : 0,06 : : 100 : 6 The Rate *per Cent.* required.

he Firſt Three Queſtions may be much more eaſily per- d by the following Table, which is only the Amounts of Pound for Thirty Nine Years.

That

That is, of $R \cdot RR \cdot RRR \cdot R^4 \cdot R^5 \cdot$ and ſo on to

Years = t.	The Amounts of 1*l.* at 6 *per Cent.* &c Compound Intereſt.	Years = t.	The Amounts of 1*l.* at 6 *per Cent.* &c. Compound Intereſt.	Years = t.	The Amoun of 1*l.* at 6 *p Cent.* &c. Co poundIntere
1	1,06 = R	14	2,2609039557	27	4,82234594
2	1,1236 = RR	15	2,3965581931	28	5,1116866
3	1,191016 = R^3	16	2,5403516847	29	5,4183878
4	1,26247696	17	2,6927727857	30	5,7434911
5	1,3382255776	18	2,8543391529	31	6,0881006
6	1,4185191122	19	3,0255995021	32	6,4533866
7	1,5030302590	20	3,2071354722	33	6,8405898
8	1,5938480745	21	2,3995636005	34	7,2510252
9	1,6894789590	22	3,6035374166	35	7,6860867
10	1,7908476965	23	3,8197496616	36	8,1472519
11	1,8982985583	24	4,0489346413	37	8,6360871
12	2,0121964718	25	4,2918707197	38	9,1542523
13	2,1329282601	26	4,5493829629	39	9,7035074

The Title of this Table ſhews its Conſtruction, and its Uſe easily appear by an Example or Two.

Example 1. *What will* 375*l*. 10*s*. *Amount to in Nine Ye at* 6 per Cent. per Annum, *&c.*

The Tabular Number againſt 9 Years is 1,689479 wh being Multiplied with the Principal 375,5 will prod 634,3993 *&c. viz.* 634*l*. 8*s*. *fere*, being the Amount Anſwer required.

Example 2. *What Principal (or Sum) muſt be put to Inte to Raiſe a Stock of* 634*l*. 8*s*. *in Nine Years Time, at* 6 Cent. per Annum, *&c.*

If the propoſed Stock, *(viz.* 634,4*)* be Divided by the bular Number that is againſt the given Number of Y *(viz.* 9.*)* the Quotient will be the Principal or (or Sum) requ *Viz.* againſt 9 is 1,689479*)*

Then 1,689479*)* 634,4*)* 375,5 = 375*l*. 10*s*. the P cipal (or Sum) as was required.

Example 3. *In what Time will* 275*l*. 10*s*. *Raiſe a Sto (or Amount to)* 634*l*. 8*s*. *at* 6 per Cent, *&c.*

Di

Divide the propoſed Stock (*viz.* 634,4) by the given Princi- (*viz.* 375,5) and the Quotient will ſhew the Tabular Num- that ſtands over againſt the Time ſought.

Thus 375,5) 634,4 (1,689479, &c. this Number being ght in the Table, will be found to ſtand againſt 9 Years, ich is the Time required.

But if the Quotient cannot be truly found in the Table of nounts for Years, as above; then take out of that Table neareſt Number that is *Leſs*, and make it a Diviſor, by ich you muſt Divide the firſt Quotient; and then ſeek the ond Quotient in the Table of Amounts for Days (*which is erted a little further on*) and it will aſſign the Number of ys. As in this Example,

In what Time will 563 *l. Amount to* 860 *l. at* 6 per Cent. Annum, *Compound Intereſt?*

Anſwer. In 7 Years and 99 Days

Thus 563) 860 (1,52753 which ſhews the Time to be more (*above*) Seven Years; For over-againſt 7 Years is 1,50363 ich being made the New Diviſor:

1,50363) 1,52753 (1,01589, &c. this Number is the areſt Amount to 99 Days.

Note, *If the Stock, Principal, and Time be given; the Rate Intereſt will be beſt found by Extracting the Root*, &c. *As fore in the Fourth Queſtion.*

The next Thing that I ſhall here propoſe, is to make this ble (*which is only Calculated for the Rate of* 6 per Cent) iverſally Uſeful for all the Rates of Compound Intereſt, *which ay preſume to ſay, is a New Improvement of my own*, be- well ſatisfied it never was Publiſhed before; and not only but I have heard ſeveral very good Artiſts affim it was im- ſible to be done.

The Method of performing it is briefly thus, Let x = the fference between 1,06 = R the Amount of 1 *l.* for one ar (in the Table) and any other propoſed Amount of 1 *l.* for e Year; which admits of Two Caſes.

Caſe 1. If the the propoſed Rate be Greater than 1,06 = R, en will $R + x$ = the true Amount of 1 *l.* for one Year at at Rate.

Case 2. But if the propoſed Rate be leſs than 1,06 =
then it will be $R - x =$ the Amount of 1 *l.* &c.

Make $\begin{cases} t-1=b \cdot t-2=c \cdot t-3=d \cdot t-4=f \&c. \\ \frac{1}{2}tb=g \cdot \frac{1}{3}cg=m \cdot \frac{1}{4}dm=n \cdot \frac{1}{5}fn=s \&c. \end{cases}$

Then will $R^t + tR^{b}x + gR^{c}xx + mR^{d}xxx$ &c. =
Amount of 1 *l.* at the given Rate, for any Time denoted by
In Caſe 1.

And $R^t - tR^{b}x + gR^{c}xx - mR^{d}xxx$ &c. = the Amou
of 1 *l.* in Caſe 2.

Which is no more but this, Let $R + x$ Or $R - x$ (whi
ſoever it) is) be Involved (as directed in *Sect.* 5. *Chap.* 2.)
the ſame Power or Height as the Index *t* the given Time
the Queſtion denotes: Rejecting all the Powers of *x* above *x*
Or *xxxx* at moſt, as uſeleſs.

Then multiply that Power of $R + x$ Or $R - x$ into
given Principal, and their Product will be the Amount
quired.

An Example or Two in each Caſe will render all Eaſy.

Example 1. *Suppoſe it were required to find what* 25
would amount to in Fifteen Years, at 8 l. per Cent. per Ann
Compound Intereſt? Here $t = 15$.

Firſt 100 : 108 : : 1 : 1,08 the Amount of 1 *l.* at 8 *per Ce*
Next 1,08 — 1,06 = 0,02 = *x*. And $R + x = 1,08$ As in Caſe
Then $R^{15} + 15R^{14}x + 105R^{13}xx + 455R^{12}xxx$ &c. =
Amount of 1 *l.* for 15 Years, at 8 *per Cent.*

Here $x = 0,02$. $xx = 0,0004$. and $xxx =$,000
By the Table $R^{15} = 2,396$
And $\begin{cases} 15R^{14}x = 2,260904 \times 15 \times ,02 = 0,678 \\ 105R^{13}xx = 2,132928 \times 105 \times ,0004 = 0,089 \\ 455R^{12}xxx = 2,012196 \times 455 \times ,000008 = 0,007 \end{cases}$
Sum = 3,171

Then 3,171736 × 256 = 812,964416 =
That is, 811 *l.* 9 *s.* 3*d* ½ *fere* Which is the Anſwer as was
quired.

Example 2. *What will* 365 *l. Amount to in Seven Y*
at Four and a half per Cent, &c.

Firſt 100 : 1,045 : : 1 : 104,5 the Amount of 1 *l.* at
per Cent.

Next 1,06 — 1,045 = 0,15 = x. Conſequently $R - 1 . x = 1{,}045$

in Caſe 2.

Then $R^7 = 1 - 7R^6x + 21R^5xx - 35R^4xxx$ &c. = the mount of 1 *l.* for 7 Years, at $4\frac{1}{2}$ *per Cent.*

ere $x = ,015$. $xx = ,000225$. *and* $xxx = ,0000033375$

By the Table $R^7 = +1{,}503630$

And $\left\{\begin{array}{l} -7R^6x = -0{,}148944 \\ +21R^5xx = +0{,}006323 \\ -35R^4xxx = -0{,}000141 \end{array}\right.$

$R^7 - 7R^6x + 21R^5xx - 35R^4xxx = 1{,}360868$

hen $1{,}360868 \times 365 = 496{,}71682 = A$.

hat is 496 *l.* 14 *s.* 3¼ *d.* is the Anſwer required.

If the *Reaſon* of theſe two *Operations* be but well underſtood, will be very eaſy to Conceive how to find P, the *Principal*, having A, t, and x given (becauſe R and its Powers are ways given by the Table.)

r $R^t \pm tR^bx + gR^cxx \pm mR^dxxx \times P = A$ (as above.)

herefore $\frac{A}{R^t \pm tR^bx + gR^cxx \pm mR^dxxx} = P$

Or if A, P, *and* t, be given, x may be found.

r $R^t \pm tR^bx + gR^cxx \pm mR^dxxx = \frac{A}{P}$ This *Æquation*

ing *Solved*, (as in *Chap.* 10.) the Value of x will be found; d then either $R + x$, Or $R - x$ will ſhew the *Rate* of *Intereſt*, &c.

But I ſhall leave the *Numerical Operations* to the *Learner's* actice, ſuppoſing enough done to ſhew how all Queſtions of kind that are limited by whole Years may be Computed.

And if the *Time* given or ſought be not *Terminated* by whole rs, but by *Weeks*, *Months*, *Quarters* or *Half-Years*, &c. for olving ſuch *Queſtions*, the beſt Way will be to *Reduce* thoſe rts of a *Year* into *Days*; that done, find an *Anſwer* accordg to the Demand of the *Queſtion*. (and agreeing to 1 *l.* as beore) for thoſe *Number* of Days; and in order to that, it will requiſite to find the *Amount* of 1 *l.* for one Day (*as in my mpendium of Algebra*, Page 110) which I ſhall here inſert.

a = the Amount ſought, then it will be

$a : a\,a :: a\,a : a\,a\,a :: a\,a\,a : a\,a\,a\,a$ &c. to a^{365}.

That is { *As one Pound is to its Amount for one Day : : is that Amount : To the Amount of Two Days : : so is that of Two Days : To that of Three Days. So on in ∺ to 365 Days.*

Then the last of the Terms will be $a^{365} = 1,06$

Put	1	$r + e = a$. And let $r = 1$
1 ⊙ 365	2	$r^{365} + 165r^{364}e + 66430r^{363}ee = a^{365} - 1,0$
2, in Num.	3	$1 + 365e + 66430ee = 1,06$
3 — 1	4	$365e + 66430ee = 0,06$
4 ÷ 66430	5	$,00549e + ee = 0,0000009032 = D$
5 ÷	6	$e = \frac{D}{,00549 + e}$

Operation ,00549) 0,0000009032 (,00016 = e
$+ e =$,00016) 55

1. *Divisor* ,0055 3532 First $r = 1$
2. *Divisor* ,00565 3390 $+ e = 0,00016$

$r + e = 1,00016$

New $r +$ 1,00016 for a *Second Operation*. Then

2, in Numb.	7	$1,06013401407 + 386,887e + 70402,172$ $= 1,06$ Hence it appears that $r - e =$
Therefore	8	$1,06013401407 - 386,887e + 70402,172$ $= 1,06$
8 ±	9	$386,887e - 70402,172ee = 0,00013401$
9 ÷	10	$,0054953 - ee = ,0000000019035503$
10 ÷	11	$e = \frac{,0000000019035503}{,0054953 - e}$

Operation ,0054953) ,0000000019035503 (,0000003 =
$- e =$,0000003 164580

Divisor ,0054950 255050 (,00000004
219800

Last $r = 1,00016$ 352503
$- e = 0,0000003464$ 329700

$r - e = a = 1,0001596536$ 228030
219800

Which being farther pursued to a *Third Operation*, it w
be $a = 1,000159653587453$ &c.

Th

This Value of *a* is the *Amount* of 1 *l.* for one Day from which, if 1 *l.* be *Subſtracted*, the *Remainder* =,0001596535 87 &c. will be the *Intereſt* of 1 *l.* for one Day. Conſequently, if any propoſed *Principal* be *Multiplied* into either of theſe, the reſpective *Product* will be the *Amount* or *Intereſt* of that *Principal* for one Day, at 6 *per Cent*, &c.

And that the *Amount (or Intereſt)* of any *Principal* or *Sum* may be eaſily computed for any *Number* of Days *Leſs* than a Year; I have here inſerted the following *Table*, which with a great deal of Care *(and I believe Exactneſs)* is Calculated from the Laſt found (1,0001596535 87453) *Amount* of 1 *l.* for one Day. To which alſo is *Annexed* a *Table* of the *Amounts* of 1 *l.* for *Months*.

Days	Amounts of 1 *l.* &c.	Days	Amounts of 1 *l.* &c.	Days	Amounts of 1 *l.* &c.
1	1,0001596536	26	1,0041592879	51	1,0081749166
2	1,0003193326	27	1,0043196055	52	1,0083358753
3	1,0004790372	28	1,0044799487	53	1,0084968597
4	1,0006387673	29	1,0046403175	54	1,0086578699
5	1,0007985229	30	1,0048007120	55	1,0088189057
6	1,0009583039	31	1,0049611320	56	1,0089799673
7	1,0011181105	32	1,0051215776	57	1,0091410545
8	1,0012779426	33	1,0052820488	58	1,0093021675
9	1,0014378002	34	1,0054425457	59	1,0094633062
10	1,0015976834	35	1,0056030682	60	1,0096244707
11	1,0017575920	36	1,0057636164	61	1,0097856608
12	1,0019175262	37	1,0059241901	62	1,0099468767
13	1,0020774859	38	1,0060847895	63	1,0101081184
14	1,0022374712	39	1,0062454148	64	1,0102693858
15	1,0023974820	40	1,0064060653	65	1,0104306789
16	1,0025575184	41	1,0065667416	66	1,0105919978
17	1,0027175803	42	1,0067274436	67	1,0107533424
18	1,0028776677	43	1,0078881712	68	1,0109147128
19	1,0030377808	44	1,0070489245	69	1,0110761090
20	1,0031979193	45	1,0072097035	70	1,0112375309
21	1,0033580850	46	1,0073705082	71	1,0113989786
22	1,0035182732	47	1,0075313385	72	1,0115604521
23	1,0036784885	48	1,0076921945	73	1,0117219513
24	1,0038387294	49	1,0078530762	74	1,0118834764
25	1,0039989958	50	1,0080139835	75	1,0120450272

Days	Amounts of 1l. &c.	Days	Amounts of 1l. &c.	Days	Amounts of 1l. &c.
76	1,0122066038	116	1,0186908655	156	1,0252166658
77	1,0123682062	117	1,0188535031	157	1,0253803457
78	1,0125298344	118	1,0190161667	158	1,0255440509
79	1,0126914885	119	1,0191788563	159	1,0257077817
80	1,0128531683	120	1,0193415719	160	1,0258715406
81	1,0130148739	121	1,0195043134	161	1,0260353247
82	1,0131766054	122	1,0196670809	162	1,0261991349
83	1,0133383627	123	1,0198298745	163	1,0263629713
84	1,0135001458	124	1,0199926934	164	1,0265268338
85	1,0136619547	125	1,0201555389	165	1,0266907225
86	1,0138237895	126	1,0203184110	166	1,0268546374
87	1,0139856501	127	1,0204813084	167	1,0270185784
88	1,0141475365	128	1,0206442319	168	1,0271825456
89	1,0143094488	129	1,0208071814	169	1,0273465389
90	1,0144713869	130	1,0209701569	170	1,0275105585
91	1,0146333511	131	1,0211331585	171	1,0276746046
92	1,0147953408	132	1,0212961861	172	1,0278386764
93	1,0149573565	133	1,0214592397	173	1,0280027746
94	1,0151193981	134	1,0216223193	174	1,0281668989
95	1,0152814655	135	1,0217854250	175	1,0283310494
96	1,0154435589	136	1,0219485567	176	1,0284952262
97	1,0156056781	137	1,0221117144	177	1,0286594291
98	1,0157678232	138	1,0222748982	178	1,0288236583
99	1,0159299941	139	1,0224381081	179	1,0289879137
100	1,0160921910	140	1,0226013440	180	1,0291521963
101	1,0162544138	141	1,0227646060	181	1,0293165051
102	1,0164166624	142	1,0229278940	182	1,0294808372
103	1,0165789370	143	1,0230912081	183	1,0296451975
104	1,0167412375	144	1,0232545483	18	1,0298095841
105	1,0169035638	145	1,0234179146	185	1,0299739969
106	1,0170659161	146	1,0235813069	186	1,0301384359
107	1,0172282944	147	1,0237447253	187	1,0303029012
108	1,0173906985	148	1,0239081699	188	1,0304673918
109	1,0175531286	149	1,0240716405	189	1,0306319206
110	1,0177155846	150	1,0242351372	109	1,0307964557
111	1,0178780665	151	1,0243986600	191	1,0309610251
112	1,0180405744	152	1,0245622089	192	1,0311256216
113	1,0182031083	153	1,0247257830	193	1,0312902445
114	1,0183656680	154	1,0248893851	194	1,0314548937
115	1,0185282578	155	1,0250530124	195	1,0316195692

Day

Days	Amounts of 1l. &c.	Days	Amounts of 1l. &c.	Days	Amounts of 1l. &c.
196	1,0317842709	236	1,0383939484	276	1,0450459680
197	1,0319489990	237	1,0385597318	277	1,0452128143
198	1,0321137534	238	1,0387255415	278	1,0453796853
199	1,0322785341	239	1,0388913778	279	1,0455466584
200	1,0324433410	240	1,0390572405	280	1,0457135092
201	1,0326081742	241	1,0392231298	281	1,0458804611
202	1,0327730339	242	1,0393890454	282	1,0460474397
203	1,0329379198	243	1,0395549876	283	1,0462144449
204	1,0331028321	244	1,0397209563	284	1,0463814768
205	1,0332677706	245	1,0398869515	285	1,0465484353
206	1,0334327355	246	1,0400529732	286	1,0467156206
207	1,0335977268	247	1,0402190214	287	1,0468827325
208	1,0337627444	248	1,0403850961	288	1,0470498711
209	1,0339277883	249	1,0405511973	289	1,0472170363
210	1,0340928586	250	1,0407173250	290	1,0473842283
211	1,0342579552	251	1,0408834793	291	1,0475514469
212	1,0344230782	252	1,0410496601	292	1,0477186923
213	1,0345882275	253	1,0412158674	293	1,0478859643
214	1,0347534033	254	1,0413821012	294	1,0480532631
215	1,0349186054	255	1,0415483616	295	1,0482205885
216	1,0350838338	256	1,0417146485	296	1,0483879407
217	1,0352490887	257	1,0418809620	297	1,0485553196
218	1,0354143699	258	1,0420473021	298	1,0487227252
219	1,0355796775	259	1,0422136687	299	1,0488901576
220	1,0357450115	260	1,0423800618	300	1,0490576166
221	1,0359103719	261	1,0425464815	301	1,0492251025
222	1,0360757587	262	1,0427129278	302	1,0493926150
223	1,0362411719	263	1,0428794007	303	1,0495601543
224	1,0364066116	264	1,0430459001	304	1,0497277204
225	1,0365710776	265	1,0432124261	305	1,0498953132
226	1,0367375701	266	1,0433789787	306	1,0500629327
227	1,0369030889	267	1,0435455579	307	1,0502305790
228	1,0370686342	268	1,0437121637	308	1,0503982521
229	1,0372342059	269	1,0438787961	309	1,0505659519
230	1,0373998041	270	1,0440454551	310	1,0507336786
231	1,0375654287	271	1,0442121407	311	1,0509014320
232	1,0377310798	272	1,0443788529	312	1,0510692121
233	1,0378967573	233	1,0445455918	313	1,0512370191
234	1,0380624612	274	1,0447123572	314	1,0514048529
235	1,0382241916	275	1,0448791493	315	1,0515727134

Days

Days	Amounts of 1 *l.* &c.	Days	Amounts of 1 *l.* &c.	Days	Amounts of 1 *l.* &c.
316	1,0517406008	339	1,0556094165	362	1,0594924636
317	1,0519085150	340	1,0557779484	363	1,0596616154
318	1,0520764559	341	1,0559465071	364	1,0598307942
319	1,0522444237	342	1,0561150927	365	1,06
320	1,0524124183	343	1,0562837053		
321	1,0525804397	344	1,0564523448		
322	1,0527484880	345	1,0566210112		
323	1,0529165631	346	1,0567897045		
324	1,0530846650	347	1,0569584248		
325	1,0532527937	348	1,0571271720		
326	1,0534209493	349	1,0572959594		
327	1,0535891317	350	1,0574647472		
328	1,0537573410	351	1,0576335753		
329	1,0539255771	352	1,0578024303		
330	1,0540938401	353	1,0579713122		
331	1,0542621300	354	1,0581402211		
332	1,0544304467	355	1,0583091570		
333	1,0545987903	356	1,0584781199		
334	1,0547671608	357	1,0586471097		
335	1,0549355582	358	1,0588161265		
336	1,0551039824	359	1,0589851703		
337	1,0552724336	360	1,0591542411		
338	1,0554409116	361	1,0593233389		

Months	The Amounts of 1 *l.* at 6 per Cent. For Months.
1	1,0048675
2	1,0097587
3	1,0146738
4	1,0196128
5	1,0245758
6	1,0295630
7	1,0345744
8	1,0396103
9	1,0446706
10	1,0497556
11	1,0548654
12	1,06

The use of this *Table* is in all respects like that of the Years, in finding the *Amount* of any given Sum for any proposed Number of Days Less than a Year.

Example 1. *Suppose it were required to find the Amount of* 375 *l. for* 210 *Days at* 6 per Cent.

The *Amount* of 1 *l.* for 210 Days is 1,0340928 &c. *per Table*. Then 1,0340928 × 375 = 387,7848 &c. = 387 *l.* 15 *s.* 8 *d.* which is the *Amount* required.

And the rest of the *Variations* may be perform'd just as the *Examples* of whole Years.

But if the *Time* given consists of *Years*, and parts of a Year, As *Quarters*, *Months*, &c. Then Reduce the Odd *Time* or parts of the Year into Days; and the Answer may then be found at *Two Operations*; As in the following *Example*.

Exa

Example 2. *Suppoſe it were required to find what* 265*l. would Amount to in five Years and* 135 *Days at* 6 per Cent, *&c.*

First the Amount of 1*l.* for { 5 Years is 1,338225 *&c.* / 135 Days is 1,021785 *&c.*

Then 1,338225 × 1,021785 × 265*l.* = 362,355232 *&c.* being the *Amount* or *Anſwer* required.

Or, if the *Amount* and *Time* are given, To find the *Principal*; Then *Multiply* the *Amount* of 1*l.* for the *Years*, and the *Amount* of 1 *l.* for the odd Days together; And by their *Product Divide* the given *Amount*, the *Quotient* will be the *Principal* required.

Example 3. *What Principal will raiſe a Stock of* 362*l.* 7*s.* 1¼*d.* *Or* 362,355232*l. in* 5 *Years and* 135 *Days, at* 6 per Cent, *&c.*

The Amount of 1 for { 5 Years is 1,338225 *&c.* / 135 Days is 1,021785 *&c.*

Then 1,338225 × 1,021785 = 1,367378 &c. the *Diviſor.*
Next 1,367378) 362,355232 = *A* (265 *l.* the *Principal* required.

Again, if the *Principal* and its *Amount* are given, To find the *Time*, at 6 *per Cent*, &c. you muſt divide the *Amount* by its *Principal*, and then proceed as in the *Third Example* Page 256 for the *Anſwer* required.

But if the *Amount* and its *Principal*, with the *Time* of its being an *Intereſt* are given, To find the *Rate* of *Intereſt*; Then proceed as in the Fourth *Queſtion* Page 255 *&c.*

Now in order to make this *Table* of *Amounts* for Days, uſeful for all *Rates* of *Intereſt*, (*as before in that for Years*) you muſt firſt find the Simple *Intereſt* of 1*l.* for one Day, both at the given *Rate*, and alſo at 6 *per Cent.* And call their Difference *x.*

Thus, ſuppoſe the given *Ratio* were 8 *per Cent. per Annum.*
First 100 : 8 : : 1 : 0,08 And 100 : 6 : : 1 : 0,06 the Two Simple *Intereſts* for one Year.

Then 365) 0,08 (,00021917 *&c.* the *Simple Intereſt* of 1*l.* for one Day at 8 *per Cent.*

And 365) 0,06 (0,00016438 *&c.* the *Simple Intereſt* of 1*l.* for one Day at 6 *per Cent.*

Their *Difference* 0,00005479 = *x* which may do indifferently well for ordinary ſmall *Queſtions*; But where Exactneſs is required, it will be convenient to make Uſe of this *Proportion.*

Viz. { As the Simple Intereſt of 1*l.* for one Day at 6 *per Cent*: Is to the Tabular Intereſt of 1*l.* for one Day : : So is the Simple Intereſt of 1*l.* for one Day, at any given Rate : To a Fourth Number.

That is, 0,00016438 : 0,00015965 : : 0,00021917 : 0,00021286
Then 0,00021286 — 0,00015965 = 0,00005321 = x.

This x being *Involved* with the reſpective *Amounts* for Days, in the ſame Manner as was done with thoſe for *Years* (*vide* Page 258) the Reſult will be the Anſwer to the *Queſtion*.

Sect. 2. Annuities or Penſions *in Arrear*; *Computed at Compound Intereſt.*

When *Annuities*, &c. are ſaid to be in *Arrear*, ſee Page 248. And I ſhall here make uſe of the ſame Letters to repreſent the ſame thing as before in that Page, ſave only that R is here equal the *Amount* of 1*l.* as in *Section* 1. of this *Chap.*

Suppoſe u = the Firſt *Years Rent* of any Annuity without *Intereſt*.

Then will $Ru+u$ = { the *Amount* of the Firſt *Years Rent*, and its *Intereſt*; More the 2d *Years Rent*.

And $RRu+Ru+u$ = { the *Amount* of the 1ſt and 2d *Years Rents*, with their *Intereſts*; More the 3d *Years Rent*, &c.

Here $RRu+Ru+u = A$ the *Amount* of any *Yearly Rent* or *Annuity*, being forborn *Three Years*. And from hence may be deduced theſe *Proportions*.

Viz. $u : Ru : : Ru : RRu : : RRu : RRRu$ and ſo on in ſo any *Number* of *Terms* or *Years* Denoted by t wherein the Laſt Term will always be uR^{t-1}

Conſequently $A - uR^{t-1}$ = the Sum of all the Antecedents
And $A-u$ = the Sum of all the Conſequents in the Series.
And therefore it would be $u : uR : : A-uR^{t-1} : A-u$
Vide Page 188.

Ergo $Au-uu = RuA-uuR^{t}$ which being Divided all by u will become $A-u = RA-uR^{t}$.

From this Laſt *Æquation* it will be eaſy to Raiſe the following *Theorems*.

Theorem 1. $\frac{uR^{t}-u}{R-1} = A$ *Theorem* 2. $\frac{RA-A}{R^{t}-1} = u$.

Theore

Theorem 3. $\left\{\frac{RA+u-A}{u} = R^t\right.$. If this *Æquation* be continually *Divided* by R, until nothing *remain*, the *Number* of thoſe *Diviſions* will be t. *See Page* 255.

4. Again $\left\{\frac{A}{u}R - R^t = \frac{A-u}{u}\right.$ If this *Equation* be Reſolved into *Numbers*, according to the Method propoſed in *Sect.* 3. *Chap* 10. the *Root* will ſhew the *Value* of R.

Queſtion 1. *If* 30*l. Yearly Rent, or Annuity*, &c. *be forborn* (*viz. remain unpaid*) *Nine Years; what will it amount to, at* 6. *per* Cent. per Annum *Compound Intereſt?*

Here is given $u=30$ $t=9$ And $R=1{,}06$ To find A. per *Theorem* 1.

$R^9 = 1{,}689479$ By the Table of Amounts for Years

$30 = u$

$R^9u = 50{,}684370$

$-u = 30,$

$R-1 = 0{,}06)\ 20{,}684370\ (344{,}7395 = 344l.\ 14s.\ 9\frac{1}{2}d. = A$ the Amount required.

Queſtion 2. *What Yearly Rent or Annuity*, &c. *being Forborn or unpaid Nine Years, will raiſe a Stock of* 344*l.* 14*s.* 9½*d.* =344,7395 *at* 6 per Cent. &c.

Here is given $A=344{,}7395$ $t=9$. And $R=1{,}06$ To find u. per *Theorem* 2.

$AR = 344{,}7395 \times 1{,}06 = 365{,}42387$

$-A = 344{,}7395$

$R^t - 1 = 1{,}689479 - 1 = 0{,}689479)\ 20{,}68437\ (30 = u.$

Queſtion 3. *In what Time will* 30*l. Yearly* Rent, *Raiſe a Stock or Amount to* 344*l.* 14*s.* 9½*d. allowing* 6 per Cent. *for the Forbearance of Payments.*

Here is given $u = 30$ $A = 344{,}7395$ And $R = 1{,}06$ To find t. Per *Theorem*, 3.

Firſt $AR + u - A = 365{,}42387 + 30 - 344{,}7395 = 50{,}68437$ And $u = 30)\ 50{,}68437\ (1{,}689479 = R^t$. Then $R=1{,}06)\ 1{,}689479\ (1{,}593848$ And $1{,}06)\ 1{,}593848\ (1{,}50363$ and ſo on until it become $1{,}06)\ 1{,}06\ (1.$ which will be at the *Ninth Diviſion*; therefore $t = 9$.

Or $R^t = 1{,}689479$ being ſought in the *Table* of *Amount* for *Years*, will be found to ſtand Over againſt 9 *Years*, which is the *Time required*.

Queſtion 4. *If* 30*l.* per Annum, *being unpaid Nine Years will Amount to* 344*l.* 14*s.* 9½*d. Allowing Compound Intereſt for every Payment as it becomes Due, What muſt the Rate of Intereſt be* per Cent. *&c.*

Here is given $u = 30$ $A = 344{,}7395$ And $t = 9$ To find R by the Laſt of the Four *Æquations*, *Viz.* $\left\{\frac{A}{u}R - R^t = \frac{A-u}{u}\right.$

Firſt $\frac{A}{u} = \frac{344{,}7395}{30} = 11{,}491317$ And $\frac{A-u}{u} = 10{,}491317$

Hence there is this *Æquation* $11{,}491317R - R^9 = 10{,}491317$

Let	1	$r + e = R$ And ſuppoſe $r = 1$
1 & 9	2	$r^9 + 9r^8e + 36r^7ee = R^9$
1, in Numb.	3	$11{,}491317e = 11{,}491317e = 11{,}491317R$
2, in Numb.	4	$1{,}000000e + 9{,}000000e + 36ee = R^9$
3 − 4	5	$10{,}491317 - 2{,}491317e - 36ee = 10{,}491317$
6 +	6	$36ee = 2{,}491317e$
$6 \div 36e$	7	$e = 0{,}06$ &c.

Firſt $r = 1$ + $e = 0{,}06$ $\Big\} = 1{,}06 = R$ { *As may be eaſily Try'd by Involving it, and Ordering it, as the Æquation above directs.*

Sect. 3. *To find the* preſent Worth *of Annuities, Penſions, Or Leaſes,* &c. *at Compound Intereſt.*

Let P = the *preſent Worth* of any *Annuity*, or *Leaſe*, &c. and the Reſt of the Letters as before.

Then, from what has been ſaid in *Section* 3. *Chap.* 11. about *Purchaſing* of *Annuities*, &c. at Simple *Intereſt*, it will be eaſy to Form the like *Theorems* here at *Compound Intereſt viz.* by Combining *Theorem* 1. *Page* 269. And *Theorem* 1. *Page* 254. into one *Theorem.*

For $\left\{\frac{uR - u}{R - 1} = A\right.$ { *The Amount of any Yearly Rent being Unpaid any Number of Years. Per* Theorem 1. *of the Laſt Section.*

And $PR = A$ { *The Amount of any Principal or Sum being put to Intereſt, for the ſame Number of Years. Per* Theorem 1. Page 254.

Hence

ence it follows, That $PR^t = \frac{uR^t}{R} - \frac{u}{1}$

z $PR^t + 1 - PR^t = uR^t - u$ being the very ſame *Æquation* h that in my *Compendium* of *Algebra Page* 112. which is there iſed from the Conſideration of Purchaſing *Annuities*, or Taking *Leaſes*, &c. to be grounded upon a *Rank* or *Series* of *Geometri-* *Proportionals* Continually Decreaſing. Thus $\frac{u}{R}$ is the *Firſt* *Greateſt Term*; R the Common *Ratio* of all the *Terms*; d P is the *Sum* of all the *Series*.

at is, $\frac{u}{R} : \frac{u}{RR} :: \frac{u}{RR} : \frac{u}{RRR} :: \frac{u}{RRR} : \frac{u}{R^4} :: \frac{u}{R^4} : \frac{u}{R^5}$ &c. *in* ∺ un- the Laſt *Term* $= \frac{u}{R^t}$ Then will $P - \frac{u}{R^t}$ be the *Sum* of all the *Antecedents*. And $P - \frac{u}{R}$ will be the *Sum* of all the *Conſe-* *nts*. Therefore it will be

$\frac{u}{RR}$. Or (in the ſame Ratio) $u : \frac{u}{R} :: P - \frac{u}{R^t} : P - \frac{u}{R}$ which produces $PR^{t+1} - uR^t = PR^t - u$. As above. om this *Æquation* may be Deduced the following *Theorems*.

Theorem 1. $\left\{ \frac{u - \frac{u}{R^t}}{R - 1} = P \right.$. *Theorems* 2. $\left\{ \frac{\overline{PR^t \times R} - PR^t}{R^t - 1} = u \right.$

Theorem 3. $\left\{ \frac{u}{P + u - PR} = R^t \right.$ *Which being Continually Divided by R, will give t.*

Theorem 4. $\left\{ \frac{u}{P} = \frac{u}{P} R^t + R^t - R^{t+1} \right.$. The Reſolving of *Æquation*, will diſcover the Value of R.

Queſtion 1. *What is* 30*l. Yearly Rent, to Continue Seven* *r, worth in ready Money, allowing* 6 per Cent. *Compound* *ereſt to the Purchaſer?*

re is given $u = 30$. $t = 7$. *And* $= 1,06$ To find P. *Theorem* 1. *Viz.* $\frac{u}{R} = \frac{30}{1,50363} = 19,9517$.

d $30 - 19,9517 = 10,483 = u - \frac{u}{R^t}$.

Then

Then $R-1=0{,}06)\ 10{,}0483\ (167{,}4716=P=167l.\ 9s.\ 5$ being the Anſwer required.

Queſtion 2. *What Annuity or Yearly Rent, to Continue Seven Years, may be Purchaſed for* 167*l.* 9*s.* 5*d. allowing* 6 per Cent *Compound Intereſt to the Purchaſer ?*

In this *Queſtion* there is given $P=167{,}4716$. $t=7$
And $R=1{,}06$ To find u . By the Second *Theorem.*
Firſt $PR^t\times R=251{,}8153\times 1{,}06=266{,}9242$
And $-PR^t=167{,}4716\times 1{,}50363=251{,}8153$

Then $R^t-1=0{,}50363)\ 15{,}1089\ (30=u$
That is $u=30l.$ the *Anſwer required.*

Queſtion 3. *How long may One have a Leaſe of* 30*l. Yearly Rent, for* 167*l.* 9*s.* 5*d. allowing* 6 per Cent. *Compound Intereſt to the Purchaſer ?*

Here is given $P=167{,}4716$. $u=30$. *And* $R=1{,}06$ To find t. By the Third *Theorem.*
Firſt $P+u=167{,}4716+30=197{,}4716$
And $-PR=177{,}5199$

Then $19{,}9515)30=u(1{,}50363=R^t$

If this $1{,}50363=R^t$ be either Continually *Divided* by $1{,}06=R$ until nothing *Remain* (As before in Page 255.) Or if it be ſought in the *Table* of *Amounts* for *Years*, &c. it will diſcover $t=7$ which is the true *Anſwer* required.

Queſtion 4. *Suppoſe One ſhould give* 167*l.* 9*s.* 5*d. for the Purchaſe of a Penſion, or Annuity of* 30*l.* per Annum, *to Continue Seven Years; At what Rate of Intereſt,* per Cent. *would the Purchaſe be made, allowing Compound Intereſt to the Purchaſer*

In this *Queſtion* there is gived, $P=167{,}4716$. $u=30$ And $t=$
To find R. Per *Theorem* 4.

The 4th *Theorem* in this *Æquation* $\left\{\frac{u}{P}=\frac{u}{P}R^t+R^t-R^t+\right.$
Which being brought into *Numbers*, and its *Root Extracted*, as in the 4th *Queſtion* of the Laſt *Section*; the Value of R will be found 1,06. *viz.* $R=1{,}06$.
And then it will be, $1:0{,}06::100:6$ the *Rate per Cent.* as was required.

The

These Four *Questions* include all the *Varieties* that can be pro-
ed about Purchasing, *Annuities* or *Leases*, &c. which are to
either immediately Enter'd upon, or in Possession at the *Time*
en the *Purchase* is made.

But such *Questions* as relate to *Annuities*, or a Taking of
ses, &c. in *Reversion*, must be *Parted* or *Divided* into *Two*
stinct Questions, each to be separately Consider'd by it self
e Page 252.) As in the following *Example*.

Examaple 1. *Suppose it were required to Compute the present*
rth of 75*l. Yearly Rent, which is not to Commence or be Enter'd*
on, until Ten Years hence; and then to continue Seven Years
er that Time: at 6 per Cent. *&c. Compound Interest?*

The First Work in this *Question*, is to find what 75*l. per An-*
m, to continue *Seven Years*, is Worth in ready Money: as if
were to be immediately enter'd upon: And to perform that,
re is given $u = 75$ $R = 1{,}06$ And $t = 7$. To find *P*. As
the First *Question* of this *Section*.

hus, $\frac{u}{R^t} = \frac{75}{150363} = 49{,}8793$ And $75 - 49{,}8793 = 25{,}1207$

$u - \frac{u}{R}$

hen, $R - 1 = 0{,}06)\ 25{,}1207 = 418{,}6783 = 418l.\ 14s.\ 6\frac{3}{4}d.$
Answer to the First Part of the *Question*.

Then the next Work will be, to find what *Principal* or *Sum*
ing put out Ten *Years*, at 6 *per Cent*, &c. will *Amount* to
8*l.* 13*s.* $6\frac{3}{4}d.$ Here is given $A = 418{,}6783$ $R = 1{,}06$
nd $t = 10$. To find *P*. Per *Theorem* 2. Page 254.

hus $R^{10} = 1{,}790847)418{,}6783 = A(233{,}7884 = 233l.\ 15s.\ 9d.$
present Worth of 75*l. per Annum* in *Reversion*, &c. As was
quired.

Example 2. *What Annuity or Yearly Rent to be Entered*
on Ten Years hence, and then to continue Seven Yeasr, may
purchased for 233*l.* 15*s.* 9*d. Ready Money, at* 6 per Cent,
. Compound Interest?

In the 1st Work of this *Question* there is given, $P = 233{,}7884$.
$= 1{,}06$. And $t - 10$ *(the Time which the Annuity is not to*
Enter'd upon) To find *A*. Per *Theorem* 1. Page 254.

hus, $PR^t = 233{,}7884 \times 1{,}790847 = 418{,}6783 = A$ the
Amount

Amount of 233*l.* 15*s.* 9*d.* put to *Interest Ten Years*, a *per Cent.* &c. Then for the Second Work of the *Que* there is given $P = 418,6783$. $R = 1,06$. And t = *(the Time that the Annuity is to be enjoy'd)* To find Per *Theorem* 2. of this *Section.*

Thus $PR^t \times R = 418,6783 \times 1,50363 \times 1,06 = 667,3095$
$-PR^t = 418,6783 \times 1,50363 = 629,5372$

$R^t - 1 = 0,50363)$ 37,7723 (75

That is, $u = 75l.$ the *Yearly Rent* required by the *Question.*

These *Two Examples* of finding P and u do fully shew *Method* that must be used in *Resolving* the Two *General*, a indeed, the most Useful *Questions* about *Annuities* or *Le* in *Reversion*: And if there be Occasion, either the *Rate*, or *Time*, viz R or t, may be found by a due Application of th respective *Theorems.*

Note, *That which hath been done in the Two Last, Sectio about Annuities or Yearly Rents, &c. at* 6 per Cent. *may also done for any Rate of Interest, by applying the Difference of Rates* (viz. *x*) *As directed in the First Section of this Chapte*

Now because that *Rents* and *Annuities*, &c. are usually p either by *Quarterly* or by *Half Yearly Payments*, and the Meth of Computing them by the Pen, may be thought a little Troub some; I have inserted the following Tables of the *Amounts* 1*l.* for Each, at 6 *per Cent.*

$= t$ Half Years	Annuities of 1*l.* at 6 *per Cent.* &c. Compound Interest.	$= t$ Half Years	Amounts of 1*l.* at 6 *per Cent.* &c. Compound Interest.	$= t$ Half Years	Amounts of 1 at 6 *per Ce* &c. Compou Interest.
1	1,0295630141	11	1,3777875592	21	1,8437905
2	1,06	12	1,4185191122	22	1,8982985
3	1,0613367949	13	1,4604548127	23	1,9544179
4	1,1236	14	1,5036302590	24	2,0121964
5	1,1568170026	15	1,5480821017	25	2,0716830
6	1,191016	16	1,5938480745	26	2,1329282
7	1,2262260228	17	1,6409070276	27	2,1959840
8	1,262 47696	18	1,6894789589	28	2,2609039
9	1,2997995842	19	1,7394250493	29	2,3277430
10	1,3382255776	20	1,7908476965	30	2,3965581

Qu

Quarterly Amounts.

Quarters of a Year = 1.	Amounts of 1 l. at 6 per Cent. &c. Compound Interest.	Quarters of a Year = 1.	Amounts of 1 l. at 6 per Cent. &c. Compound Interest.	Quarters of a Year = 1.	Amounts of 1 l. at 6 per Cent. &c. Compound Interest.
1	1,01467 38461	21	1,35780 24938	41	1,81712 63199
2	1,02956 30141	22	1,37778 75592	42	1,84379 05523
3	1,04467 06634	23	1,39800 50019	43	1,87084 60509
4	1,06	24	1,41851 91122	44	1,89829 85583
5	1,07553 42769	25	1,43933 42435	45	1,92615 38989
6	1,09133 67949	26	1,46045 48127	46	1,95441 79853
7	1,10735 09032	27	1,48188 53020	47	1,98309 68140
8	1,1236	28	1,50363 02590	48	2,01219 64718
9	1,14008 75335	29	1,52569 42978	49	2,04172 31330
10	1,15681 70026	30	1,54808 21017	50	2,07168 30644
11	1,17379 19574	31	1,57079 84203	51	2,10208 26228
12	1,191016	32	1,59384 80745	52	2,13292 82601
13	1,20849 27856	33	1,61723 59557	53	2,16422 65211
14	1,22622 60228	34	1,64096 70276	54	2,19598 40483
15	1,24421 94748	35	1,66504 63253	55	2,22820 75801
16	1,26247 696	36	1,68947 89589	56	2,26090 39557
17	1,28100 23527	37	1,71427 01133	57	2,29408 01123
18	1,29979 95842	38	1,73942 50493	58	2,32774 30912
19	1,31887 26433	39	1,76494 91048	59	2,36190 00349
20	1,33822 55776	40	1,79084 76965	60	2,39655 81931

Either of these *Tables* may also be made Useful for any proposed *Rate* of *Interest*; by making the $\frac{1}{2}$ or $\frac{1}{4}$ of the *Difference* of the *Rate* $= x$, &c.

As for Instance, Suppose any of the aforesaid *Questions* about *Annuities* or *Rents*, &c. were to be computed at 8 *per Cent. per Annum.*

Then 1,08 − 1,06 = 0,02 = x for *Yearly Payments*; as before

Consequently 2) 0,02 (0,01 = x for *Half Years Payments.*

Or 4) 0,02 (0,005 = x for *Quarterly Payments.*

Now these Values of x, although they are not really true, yet they may serve indifferently well for small *Rents*; as I have already said, *Page* 265. But if you would Work exactly.

Then $\sqrt{}$ 1,08 = 1,0392304845 &c.

− $\sqrt{}$ 1,06 = 1,0295680141 *vide Table Page* 272.

Difference = 0,0096674704 = x for $\frac{1}{2}$ *Yearly Payments.*

And $\sqrt{:}\sqrt{}\ 1{,}08 = 1{,}0194263092$ &c.

$-\sqrt{:}\sqrt{}\ 1{,}08 = 1{,}0146738461$ See the *Last Table.*

Their Difference $0{,}0047524631 = x$. for *Quarterly Payments*

These are the true *Values* of x, which being *Involved* with their respective *Amounts* (as before for *Years*, &c.) according as the *Question* requires, the *Result* will be the *Answer* at 8 *per Cent.* &c. The like may be done for any other *Rate*, either *Greater* or *Less* than 6.

Now, altho' this Method (*See Page* 257, and 258, &c.) of making the *Tables* that are only Calculated for the *Rate* of 6 *per Cent.* General and Useful for all *Rate* of *Compound Interest*, be really True; yet it was rather propos'd to shew what may possibly be perform'd by the Pen, without a great many *Tables* of several *Rates*, than intended for common Practice.

For it must needs be confess'd, that *Tables* Calculated or Propose for any designed *Rate* of *Interest*, are much more Ready and Useful in common Practice. And therefore since the Legislative Power, have thought fit to reduce the *Rate* of *Interest*, and have settled it by an Act of Parliament, at 5 *per Cent*: I've therefore been at the Trouble (*which was not a little*) to calculate the following *Tables* for that *Rate*, but don't think it convenient to take the *Tables*, at 6 *per Cent.* out of the Book, because the Examples are all suited to them, and not only so but they may be found Useful in the taking of *Leases* for Houses, &c. For in those Cases, the *Purchaser* is always allowed more *Interest* for his purchase Money, than the common *Rate* paid upon the Loan of Money.

He

Here follows New *Tables* of the *Amounts* of one Pound, at the *Rate* of 5 *per Cent. per Annum* Compound Interest, for *Years, Half Years, Quarters, Months and Days.*

I. The *Table* of the *Yearly Amounts* of 1*l.* &c.

Years	The Amounts of 1*l.* &c.
1	1,05 = *R*
2	1,1025 = *RR*
3	1,157625 = R^3
4	1,21550625
5	1,27628156
6	1,34009564
7	1,40710042
8	1,47745544
9	1,55132822
10	1,62889463
11	1,71033936
12	1,79585633
13	1,88564914
14	1,97993160
15	2,07892818
16	2,18287459
17	2,29201832
18	2,40661923
19	2,52695019
20	2,65329770
21	2,78596259
22	2,92526072
23	3,07152375
24	3,22509994
25	3,38635494
26	3,55567269
27	3,73345632
28	3,92012914
29	4,11613599
30	4,32194239
31	4,53803949
32	4,76494147
33	5,00318854
34	5,25334797
35	5,51601536
36	5,79181613
37	6,08140694
38	6,38547729
39	6,70475115

II. The *Table* of the *Half Yearly Amounts* of 1*l.* &c.

Half Yrs.	The Amounts of 1*l.* &c.
1	1,02469507
2	1,05
3	1,07592983
4	1,1025
5	1,12972632
6	1,157625
7	1,18621264
8	1,21550625
9	1,24552327
10	1,27628156
11	1,30779943
12	1,34009564
13	1,37318940
14	1,40710042
15	1,44184887
16	1,47745544
17	1,51394132
18	1,55132822
19	1,58963838
20	1,62889463
21	1,66912030
22	1,71033936
23	1,75257632
24	1,79585633
25	1,84020513
26	1,88564914
27	1,93221539
28	1,97993160
29	2,02882616
30	2,07892818

III. The *Table* of the *Quarterly Amounts* of 1*l.* &c.

Quarters	The Amounts of 1*l.* &c.	Quarters	The Amounts of 1*l.* &c.	Quarters	The Amounts of 1*l.* &c.
1	1,01227223	21	1,29194439	41	1,64888480
2	1.02469507	22	1,30779943	42	1,66912031
3	1,03727037	23	1 32384905	43	1,68960414
4	1,05	24	1,34009564	44	1,71033936
5	1,06288585	25	1,35654161	45	1,73132904
6	1,07592983	26	1,37318940	46	1,75257632
7	1,08913389	27	1,39004151	47	1,77408435
8	1,1025	28	1,40710042	48	1,79585633
9	1,11603014	29	1,42436869	49	1,81789549
10	1,12972632	30	1,44184887	50	1,84020913
11	1,14359059	31	1,45954358	51	1,86278856
12	1,157625	32	1,47745544	52	1,88564914
13	1.17183164	33	1,49558712	53	1,90879027
14	1,18621264	34	1,51394132	54	1,93221535
15	1,20077012	35	1,53252076	55	1,95592795
16	1,21550625	36	1,55132822	56	1,97993160
17	1,23042323	37	1,57036648	57	2,00422975
18	1,24552327	38	1,58963838	58	2,02882611
19	1,26080862	39	1,60914680	59	2,05372437
20	1,27628156	40	1,62889463	60	2,07892810

IV. The *Table* of the *Monthly Amounts* of 1*l.* &c.

Months	The Amounts of 1*l.* &c.	Months	The Amounts of 1*l.* &c.	Months	The Amounts of 1*l.* &c.
1	1,00407412	5	1,02053728	9	1,03727070
2	1.00816485	6	1,02469507	10	1,04149600
3	1,01227223	7	1,02886981	11	1,04573900
4	1,01639636	8	1,03306155	12	1,05

NOTE: The *Amount* of one *Pound* for one *Day*, 1,00013368072 25. &c. (found as that in *Page* 260) But in following *Table*; I take only Nine of those *Figures*, as be sufficient in Practice, for Computing the *Interest* of any S not exceeding One Hundred Millions of *Pounds*.

V. T

V. The *Table* of the *Daily Amounts* of 1*l.* &c.

Days =	The Amounts of 1 *l.* &c.	Days =	The Amounts of 1 *l.* &c.	Days =	The Amounts of 1 *l.* &c.
1	1,00013368	36	1,00482378	71	1,00953587
2	1,00026738	37	1,00495810	72	1,00967082
3	1,00040109	38	1,00509245	73	1,00980579
4	1,00053483	39	1,00522681	74	1,00994079
5	1,00066858	40	1,00536119	75	1,01007579
6	1,00080235	41	1,00549558	76	1,01021083
7	1,00093614	42	1,00563000	77	1,01034587
8	1,00106994	43	1,00576443	78	1,01048093
9	1,00120377	44	1,00589888	79	1,01061602
10	1,00133761	45	1,00603335	80	1,01075112
11	1,00147147	46	1,00616784	81	1,01088623
12	1,00160535	47	1,00630234	82	1,01102137
13	1,00173924	48	1,00643687	83	1,01115652
14	1,00187315	49	1,00657141	84	1,01129169
15	1,00200708	50	1,00670597	85	1,01142688
16	1,00214103	51	1,00684055	86	1,01156209
17	1,00227500	52	1,00697514	87	1,01169732
18	1,00240899	53	1,00710975	88	1,01183256
19	1,00254299	54	1 00724438	89	1,01196783
20	1,00267701	55	1,00737903	90	1,01210311
21	1,00281105	56	1,00751370	91	1,01223841
22	1,00294510	57	1,00764839	92	1,01237372
23	1,00307918	58	1,00778309	93	1,01250906
24	1,00321327	59	1,00791781	94	1,01264441
25	1,00334738	60	1,00805255	95	1,01277978
26	1,00348151	61	1,00818731	96	1,01291517
27	1,00361565	62	1,00832208	97	1,01305058
28	1,00374982	63	1,00845687	98	1,01318600
29	1,00388400	64	1,00859168	99	1,01332145
30	1,00401820	65	1,00872651	100	1,01345691
31	1,00415242	66	1,00886136	101	1,01359239
32	1,00428665	67	1,00899623	102	1,01372788
33	1,00442091	68	1,00913111	103	1,01386340
34	1,00455518	69	1,00926601	104	1,01399893
35	1,00468947	70	1,00940093	105	1,01413448

Days	The Amounts of 1 l. &c.	Days	The Amounts of 1 l. &c.	Days	The Amounts of 1 l. &c.
106	1,01427005	146	1,01970775	186	1,02517459
107	1,01440564	147	1,01984406	187	1,02531164
108	1,01454125	148	1,01998039	188	1,02544870
109	1,01467687	149	1,02011675	189	1,02558578
110	1,01481252	150	1,02025312	190	1,02572288
111	1,01494818	151	1,02038950	191	1,02586000
112	1,01508386	152	1,02052591	192	1,02599714
113	1,01521955	153	1,02066234	193	1,02613430
114	1,01535527	154	1,02079878	194	1,02627147
115	1,01549100	155	1,02093524	195	1,02640866
116	1,01562675	156	1,02107172	196	1,02654588
117	1,01576252	157	1,02120822	197	1,02668310
118	1,01589831	158	1,02134473	198	1,02682035
119	1,01603412	159	1,02148127	199	1,02695762
120	1,01616994	160	1,02161782	200	1,02709490
121	1,01630578	161	1,02175439	201	1,02723221
122	1,01644164	162	1,02189098	202	1,02736953
123	1,01657752	163	1,02202758	203	1,02750686
114	1,01671349	164	1,02216421	204	1,02764422
125	1,01684933	165	1,02230085	205	1,02778160
126	1,01698527	166	1,02243751	206	1,02791899
127	1,01712122	167	1,02257419	207	1,02805640
128	1,01725719	168	1,02271089	208	1,02819384
129	1,01739317	169	1,02284761	209	1,02833129
130	1,01752918	170	1,02298434	210	1,02846875
131	1,01766521	171	1,02312109	211	1,02860624
132	1,01780125	172	1,02325787	212	1,02874375
133	1,01793731	173	1,02339466	213	1,02888127
134	1,01807338	174	1,02353147	214	1,02901881
135	1,01820948	175	1,02366829	215	1,02915637
136	1,01834559	176	1,02380514	216	1,02929395
137	1,01848173	177	1,02394200	217	1,02943154
138	1,01861788	178	1,02407888	218	1,02956916
139	1,01875405	179	1,02421578	219	1,02970679
140	1,01889024	180	1,02435270	220	1,02984445
141	1,01902644	181	1,02448964	221	1,02998212
142	1,01916267	182	1,02462659	222	1,03011980
143	1,01929891	183	1,02476356	223	1,03025751
144	1,01943517	184	1,02490055	224	1,03039524
145	1,01957145	185	1,02503756	225	1,03053298

Day

Days	The Amounts of 1 l. &c.	Days	The Amounts of 1 l. &c.	Days	The Amounts of 1 l. &c.
226	1,03067074	266	1,03619636	306	1,04175160
227	1,03080852	267	1,03633488	307	1,04189086
228	1,03094632	268	1,03647342	308	1,04203015
229	1,03108414	269	1,03661197	309	1,04216944
230	1,03122197	270	1,03675055	310	1,04230876
231	1,03135983	271	1,03688914	311	1,04244810
232	1,03149770	272	1,03702775	312	1,04258745
233	1,03163559	273	1,03716638	313	1,04272683
234	1,03177350	274	1,03730503	314	1,04286622
235	1,03191143	275	1,03744370	315	1,04300563
236	1,03204938	276	1,03758239	316	1,04314506
237	1,03218734	277	1,03772109	317	1,04328451
238	1,03232533	278	1,03785982	318	1,04342397
239	1,03246333	279	1,03799856	319	1,04356346
240	1,03260135	280	1,03813732	320	1,04370297
241	1,03273939	281	1,03827609	321	1,04384249
242	1,03287744	282	1,03841489	322	1,04398203
243	1,03301552	283	1,03855371	323	1,04412159
244	1,03315361	284	1,03869254	324	1,04426117
245	1,03329173	285	1,03883139	325	1,04440077
246	1,03342986	286	1,03897027	326	1,04454038
247	1,03356801	287	1,03910916	327	1,04468002
248	1,03370617	288	1,03924817	328	1,04481967
249	1,03384436	289	1,03938699	329	1,04495934
250	1,03398157	290	1,03952594	330	1,04509903
251	1,03412079	291	1,03966491	331	1,04523874
252	1,03425903	292	1,03980389	332	1,04537847
253	1,03439729	293	1,03994289	333	1,04551822
254	1,03453557	294	1,04008191	334	1,04565798
255	1,03467387	295	1,04022095	335	1,04579777
256	1,03481218	296	1,04036001	336	1,04593757
257	1,03495052	297	1,04049908	337	1,04607739
258	1,03508887	298	1,04063818	338	1,04621723
259	1,03522724	299	1,04077729	339	1,04635709
260	1,03536563	300	1,04091642	340	1,04649697
261	1,03550404	301	1,04105557	341	1,04663686
262	1,03564247	302	1,04119474	342	1,04677678
263	1,03578091	303	1,04133393	343	1,04691671
264	1,03591938	304	1,04147314	344	1,04705667
265	1,03605786	305	1,04161236	345	1,04719664

Days ℔	The Amounts of 1 *l.* &c.	Days ℔	The Amounts of 1 *l.* &c.	Days ℔	The Amounts of 1 *l.* &c.
346	1,04733663	353	1,04831708	360	1,04929845
347	1,04747664	354	1,04845722	361	1,04943872
348	1,04761666	355	1,04859738	362	1,04957901
349	1,04775671	356	1,04873756	363	1,04971932
350	1,04789677	357	1,04887775	364	1,04985965
351	1,04803686	358	1,04901797	365	1,04999999
352	1,04817696	359	1,04915820	*viz.*	1,05

I think it needléss to say any Thing of the Use of these *Tables* because I take it for granted, that whoever understands the Work of the foregoing *Examples*, at 6 *per Cent.* Cannot but know how to make use of these *Tables* at 5 *per Cent.* As Occasion requires.

Thus far concerning such *Annuities* or *Leases*, &c. that are *Limited* by any Assigned *Time*; and 'tis only such that can be computed by *Theorems* or certain *Rules*. However it may not perhaps be *Unacceptable*, to insert a brief Account of some *Estimates* that have been Reasonably made, by *Two* very ingenious *Persons*, about the Proportion, or Difference of *Mens Lives*, according to their several *Ages*; which may be of good Use in computing the *Values* of *Annuities*, or taking of *Leases* for *Lives*, &c.

Sir *William Petty* in his Discourse made before the *Royal Society* (*Anno* 1674.) concerning the Use of **Duplicate Proportion**, in the Life of *Man* and its *Duration*; saith, That its found by *Experience* there are more Persons Living of between 16 and 26 *Years Old* than of any other *Age* or *Decade* of Years in the whole Life of *Man* (*viz.* 70 or 80 *Years.*) *His Reason for that Assertion I shall omit*; but supposing it true, he thence Infers, That the *Roots* of every *Number* of *Mens Ages* under 16 (whose *Root* is 4) compared with the said Number 4 doth shew the *Proportion* of the Likelyhood of such Men reaching the *Age* of 70 Years.

As for *Example*, 'Tis 4 *Times* more likely, that One of 16 *Years Old* should live to 70, than a New-Born *Babe*: 'Tis 9 *Times* more likely, that One of 9 *Years Old* should attain the *Age* of 70, than the said *Infant*, &c.

On the other Hand. 'Tis 5 to 4, that One of 25 *Years Old* will Die before One of 16: And 6 to 5, that One of 36 will Die before One of 25. And so on according to the *Roots* of any other declining *Age*, compared with (4,6) the *Root* of 25 which is the *Year* of *Perfection* according to the Sense of our *Law*, and the *Age* for whose Life a *Lease* is most *Valuable*.

2. The

2. *The Ingenious* and Great *Mathematician*, Doctor *Edmund Halley*, (*In Philoſoph. Tranſact. Numb.* 196) doth with great *Induſtry* and *Skill*, draw an *Eſtimate* of the *Proportion* of *Mens Lives*, from the *Monthly Tables* of the *Births* and *Funerals* in *Breſlaw*, the *Capital City* of the *Province* of *Sileſia*; Or, as the *Germans* call it, *Schleſia*. Whence he proves, that its 80 to 1 a Perſon of 25 *Years Old* will not Die in a *Year*: That its $5\frac{1}{7}$ to 1, that a Man of 40 will live 7 *Years*: That a Man of 30 *Years Old* may reaſonably expect to Live 27 or 28 *Years*, &c.

Now from theſe and the like *Proportions* (he juſtly infers that) the *Price* of *Inſurance* upon *Lives* ought to be Regulated, there being a great Difference between the *Life* of a Man of 20, and One of 50. For *Example*; 'Tis 100 to 1, that of a Man of 20 Dies not in a *Year*, and but 38 to 1, for a Man 50 *Years* of *Age*. And upon theſe alſo depends the *Valuation* of *Annuities* for Lives: For it is plain, that the *Purchaſer* ought to pay only ſuch a Part of the Value of any *Annuity*, as he hath *Chances* that he is Living.

And for that Purpoſe he hath taken the Pains (*which was not little*) to compute the following *Table*, that ſhews the *Value* of *Annuities*) for every Fifth Year of *Age* to the 70th.

Age	Year's Purchaſe	Age	Year's Purchaſe	Age	Year's Purchaſe
1	10,28	25	12,27	50	9,21
5	13,40	30	11,72	55	8,51
10	13,44	35	11,12	60	7,60
15	13,33	40	10,57	65	6,54
20	12,78	45	9,91	70	5,32

The ſame *Ingenious Gentleman* proceeds on, and ſhews how to *Eſtimate* or find the *Value* of *Two Lives*, and then of *Three Lives*, which being too long a Diſcourſe to be recited here, I have, for Brevities ſake, omitted it; and ſhall only add this ſerious Obſervation.

Viz. How unjuſtly we repine at the Shortneſs of our *Lives*, and think our ſelves wrong'd if we attain not to *Old Age*; whereas it appears, that the *One Half* of thoſe that are Born, Die in Seventeen Years Time. For by the aforeſaid Bills of Mortality at *Breſlaw*, it was found, that 1238 were in that time reduced to 616. So that inſtead of Murmuring at what we call a *Short Life*, we ought to account it as a great Bleſſing that we have *Survived*, perhaps by many Years, that *Period* of *Life* whereat the one half of the whole Race of Mankind does not Arrive.

Sect. 4. *Of Purchasing* Free-hold, *Or* Real Estates *at Compound Interest.*

All *Free-hold* or *Real Estates*, are supposed to be Purchased or Bought to continue for *Ever* (viz, *without any Limited Time*); Therefore the Business of Computing the true *Value* of such *Estates* is grounded upon a *Rank* or *Series* of *Geometrical Proportionals* continually *Decreasing*, *ad Infinitum*.

Thus, Let P, u, R, Denote the same *Data* as in the last *Section*. Then the *Series* will be, $\frac{u}{R} \cdot \frac{u}{RR} \cdot \frac{u}{R^3} \cdot \frac{u}{R^4} \cdot \frac{u}{R^5}$ and so on in $\div$ until the Last *Term* $= 0$. Then will $P -$ (viz. P) be the *Sum* of all the *Antecedents*. And $P - \frac{u}{R}$ will be the *Sum* of all the *Consequents*; therefore it will be, $u : \frac{u}{R} :: P : P - \frac{u}{R}$ which produces $PR - u = P$.

This *Æquation* affords these following *Theorems*:

Theorem 1. $PR - P = u$. *Theorem* 2. $\left\{ \frac{u}{R-1} = P. \right.$

Theorem 3. $\left\{ \frac{P+u}{P} = R. \right.$

Example. *Suppose a Free-hold Estate of* 75 *l. Yearly Rent were to be Sold; what is it worth allowing the Buyer* 6 per Cent. *Compound Interest for his Money?*

In this *Question* there is given $u = 75$. $R = 1{,}06$ To find Per *Theorem* 2. Thus $R - 1 = 0{,}06) 75 = u$ (1250 *l.* = the *Answer* required. And so for any of the rest as Occasion requires. But if the *Rent* is to be paid, either by *Quarterly* Or *Half Yearly Payments*;

Then $R = \sqrt{}$ 1,06 for *Half Yearly*
And $R = \sqrt{} : \sqrt{}$ 1,06 for *Quarterly* } *Payments* at 6 *per Cent.*

Or $\left\{ \begin{array}{ll} R = & 1{,}08 \text{ for } Yearly \\ R = \sqrt{} & 1{,}08 \text{ for } Half\text{-}Yearly \\ R = \sqrt{} : \sqrt{} & 1{,}08 \text{ for } Quarterly \end{array} \right\}$ *Payments* at 8 *per Cent.*

The like is to be understood for any other proposed *Rate* of *Interest* either *Greater*, or *Less* than 6 *per Cent.*

The Application of these *Theorems* to Practice, is so very Easy, that it's needless to insert more *Examples*.

A

AN INTRODUCTION TO THE Mathematicks.

PART III.

CHAP. I.

Of Geometrical Definitions, &c.

Sect. 1. Of Lines; and Angles.

A Point *hath no Parts:* That is, a *Geometrical Point* is not any *Quantity*, but only an *Assinable Place* in any *Quantity*, denoted by a *Point*: As at *A.* and *B.*

A. *B.*

Such a Place may be conceived so Infinitely Small, as to be void of Leangth, Breadth and Thickness; and therefore a Point may be said to have no *Parts.*

2. A Line is called a *Quantity* of one *Dimension*, because it may have any supposed *Length*, but no *Breadth* nor *Thickness*, being made or Represented to the *Eye*, by the *Motion* of a *Point.*

That is, If the *Point* at *A*, be *Moved* (*upon the same Plain*) to the *Point at B*, it will describe a *Line* either *Right*, or *Circular*, (*viz. Crooked*) according to its *Motion.*

Therefore the *Ends* or *Limits* of a *Line* are *Points.*

3. A Right Line, is that *Line* which lieth *Even* or *Streight* betwixt those *Points* that *Limit* its *Length*, being the *shortest Line* that can be drawn between any *Two Points.* As the *Line A B.*

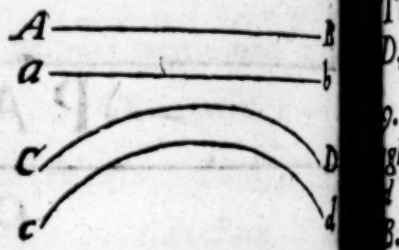
A ——— *B.*

Therefore between any Two Points, there can Lie, or be Drawn but one Right Line.

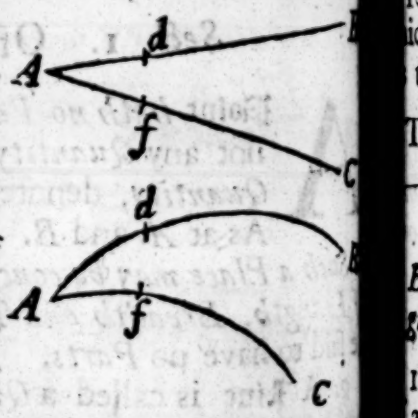
4. A **Circular**, *Crooked* or **Oblique** *Line*, is that which *Lies* bending between those *Points* which *Limit* its *Length*, as the *Lines C D*, or *F G*, &c.

Of these kinds of Lines there are various Sorts; but those of the Circle Parabola, Elipsis, and Hyperbola are of most general Use in Geometry; of which a particular Account shall be given further on.

5. **Parallel Lines**, as those that lie *Equally Distant* from one another in all their *Parts*, viz. such *Lines* as being *Infinitely extended* (*upon the same Plain*) will never *Meet*, As the *Lines A B* and *a b*, or *C D* and *c d*.



6. **Lines** not **Parallel**, but **Inclining**, (*viz. Leaning*) one towards another, whether they are *Right Lines*, or *Circular Lines* will if they are *Extended*) *Meet*, *and* make an *Angle*; the *Point* where they *Meet* is called the *Angular Point*, As at *A*. And according as such *Lines* stand, *nearer* or *further* off each other, the *Angle* is said to be *Lesser*, or *Greater*, whether the *Lines* that *Include* the *Angle* be *Long*, or *Short*. That is, the *Lines A d*, And *A f Include* the same *Angle* as *A B*, and *A C* doth; notwithstanding that *A B* is *Longer* than *A d*, &c.

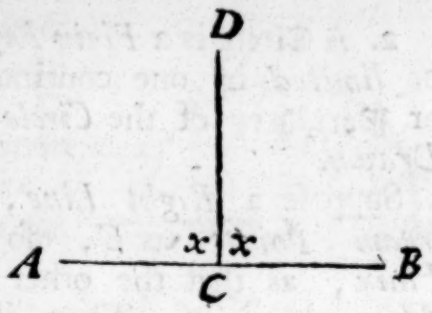


7. All **Angles** *Included* between *Right Lines*, are called *Right-lin'd Angles*; and those *Included* between *Circular Lines* are called *Spherical Angles*. But all *Angles*, whether *Right-lin'd* or *Spherical*, fall under one of these *Three Denominations*.

Viz. { A **Right Angle.** / An **Obtuse Angle.** / An **Acute Angle.** }

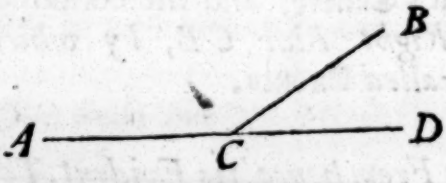
8. A **Right Angle** is that which is *Included* betwixt *Two Lines*, that *Meet* one another *Perpendicular*.

That is, When a *Right-line* D C, *Meets* with another *ght-line* as *A B*, ſo directly hat it neither *Inclines* nor *De-es* to one *Side* more than the er, but makes the *Angles* on h *Sides* of it *Equal*, as at *x*, *x*; en are thoſe *Angles* called *ht Angles*; and the *Lines* ſo *ting* are ſaid to be *Perpendicular* to each other.



That is, *A C*, aud *C B*, are *Perpendicular* to *D C*, as well *D, C* is to *either* or both of them.

9. An **Obtuſe Angle** is that which is *Greater* than a *Right gle*. Such is the *Angle Inclu-* between the *Lines A C* and *B*.



10. An **Acute Angle** is that ich is *Leſs* than a *Right Angle* the *Angle Included* between the *Lines C B* and *C D*.

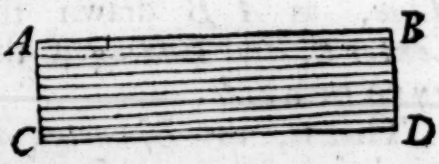
Theſe *Two Angles* are generally called **Oblique** *Angles*.

Sect. 2. Of a Circle, &c.

Before a Circle and its Parts are Defined, it will be convenient give a brief Account of Superficies in general.

1. A **Superficies** or **Surface** is the upper, or very *Out-ſide* any *viſible* Thing. But by *Superficies* in **Geometry**, is meant ly ſo much of the *Out-ſide* of any Thing as is *Incloſed* within *Line* or *Lines*, according to the Form or *Figure* of the Thing *ſigned*; and it is *produced* or *formed* by the *Motion* of a *Line*, a *Line* is deſcribed by the *Motion* of a *Point*; thus:

Suppoſe the *Line A B* were *equally oved (upon the ſame Plain)* to D; Then will the *Points* at *A* and *Deſcribe* the *Two Lines A C* and D; and by ſo doing they will *rm* (and *Incloſe*) the **Superficies** or *Figures A B C D*, be-g a *Quantity of Two Dimenſions*, viz. it hath *Length* and *eadth*, but not *Thickneſs*. *Conſequently the Bounds or Limits a Superficies are Lines.*



Note,

Note, *The Superficies of any Figure, is usually called its*

2. A Circle is a *Plain Regular Figure*, whose *Area is bound* or *limited* by one continued *Line*, called the Circumfer or Periphery of the *Circle*, which may be thus *Described Drawn.*

Suppose a *Right Line*, as *C B*, to have one of it's *tream Points* as *C*, so fixt upon any *Plain*, as that the other *Point* at *B* may *Move* about it; Then if the *Point* at *B* be *Moved Round* about (*upon the same Plain*) it will *Describe a Line equally Distant* in all its *Parts* from the *Point C*, which will be the *Circumferance* or *Periphery* of that *Circle*; the *Point C*, will be it's Center, and the contained *Space* will be its *Area*, and *Right Line C B, by which the Circle is thus Described, called* Radius.

Consectary.

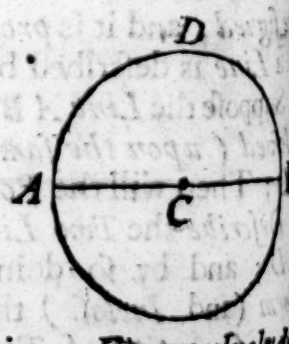
From hence 'tis Evident, That an Infinite Number of Right Lin may be drawn from the Center of any Circle to touch it's Perither which will be all equal to one another, because they are all Radius

And with a little Consideration it will be easie to conceive th no more than two equal Right Lines can be drawn from any Poi with a Circle to touch it's Periphery, but from the Center on (9. e. 3.)

3. Equal Circles are those which have *Equal Radius's*; fo its plain by the last *Definition*, that one and the same *Radi* (as *C B*) must needs *Describe Equal Circles*, how many so ver they are.

4. The *Diameter* of a *Circle*, is *Twice* its *Radius joyned* into one *Right Line*, as *A B* drawn through the *Center C*, and *Ending* at the *Periphery* on each *Side*.

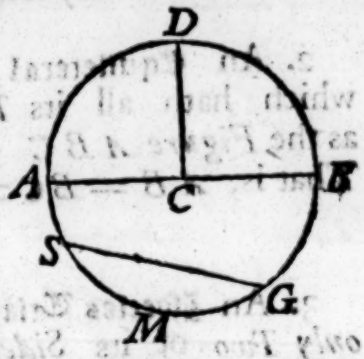
That is, the *Diameter divides* the *Circle* into *Two* equal Parts.



5. A *Semicircle* (viz. *Half a Circle*) is a *Figure Include* between the *Diameter*, and *Half* the *Periphery cut off* b the *Diameter*; As *A D B*.

6. A

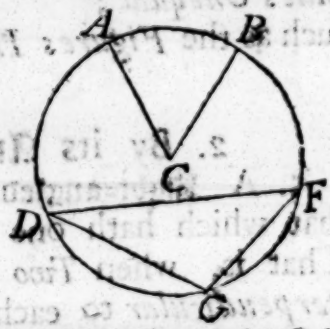
6. A **Quadrant** is *Half* a *Semicircle*, viz. one *Quarter* of a cle; and 'tis made by the *Radius* D C *ſtanding Perpendicular* up- the *Diameter* at the *Center* C, ting the *Periphery* of the *Semecircle* the *middle*, as at *D*. *Therefore a adrant, Or half the Semicircle is the eaſure of a Right Angle.*



7. A **Chord Line**, or the *Subtenſe* an *Arch*, is any *Right Line that cuts* e *Circle* into *Two unequal Parts, as the Line S G*; and is rays *Leſs* than the *Diameter.*

8. A **Segment** of a *Circle*, is a *Figure Included* betwixt the ord and that *Arch* of the *Periphery* which is *cut off* by the ord: And it may either be *Greater*, or *Leſs* than a *Semicircle*; s the *Figure S M G*, or *S D G.*

9. A **Sector** is a *Figure Included* between *Two Radius*'s of the rcle, and that *Arch* of its *Periphery* here they *Touch*, as the *Figure A C B*, nd the *Arch A B* is the *Meaſure* of e *Angle* at C, *Included* betwixt the adius's *A C*, and *B C.*



Note, *All Angles of Sectors are cal- d Angles at the Center of a Circle.*

10. An **Angle** in the *Segment* of a *Circle* is that which is ncluded between *Two Chords* that *flow* from *one* and the *ſame* oint in the *Periphery*, as at *D*, and *meet* with the *Ends* of nother *Chord Line*, as at *F* and *G.*

That is, the *Angles* at *D*, at *F*, and at *G*, are called ngles at the *Periphery*, or *Angles Standing* on the *Segment* f a *Circle.*

Sect. 3. *Of* Triangles.

There are Two Kinds of Triangles, viz. *Plain and Spherical*; ut *I ſhall not give any Definition of the Spherical, becauſe they ore immediately Relate to Aſtronomy.*

1. A **Plain Triangle** is a *Figure* whoſe *Area* is *contained* within the *Limits* of *Three Right Lines* called *Sides*, *Including Three Angles*: And it may be *Divided*, and takes it *Name* either according to its *Sides* or *Angles.*

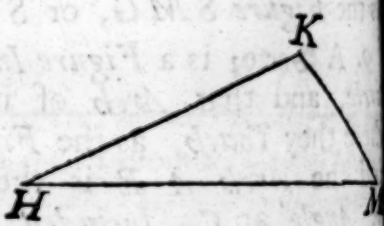
1. By

1. By its Sides.

2. An **Equilateral Triangle** is that which hath all its *Three Sides* equal, as the *Figure A B C*
That is, $AB = BC = AC$.

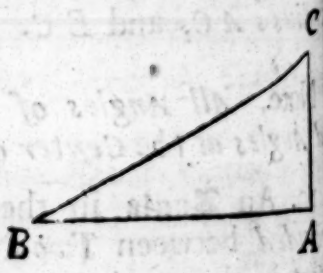
3. An **Isoceles Triangle** is that which hath *only Two* of its *Sides Equal* as the *Figure B D G* That is, $BD = DG$; but the *Third Side B G* may be either *Greater* or *Less*, as Occasions requires.

4. A **Scalenous Triangle.**
is that which hath all its *Three Sides Unequal*;
such as the *Figures H K M.*



2. By its Angles.

5. A **Right-angled** *Triangle*, is that which hath one *Right Angle*; That is, when *Two* of its *Sides* are *Perpendicular* to each other, as *C A* is supposed to be to *B A.* Therefore the *Angle* at *A*, is a *Right Angle*, Per *Defi.* 8. *Sect.* 1.



Note, *The longest Side of every Right-angled Triangle (as B C) is called Hypotenues, and the Longest of the other Two Sides which Include the Right Angle (as B A) is called Base. The Third Side (as C A) is called Cathetus or Perpendicular.*

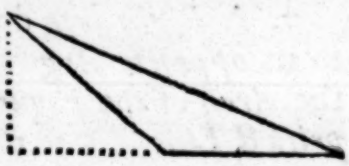
6. An **Obtuse-angled** *Triangle* is that which hath *one* of its *Angles Obtuse*, and its called an *Amblygonium Triangle.* Such is the *Third Triangle H K M.*

7. An **Acute-angled Triangle** is that which hath all its *Angles Acute*, and its called an *Oxygonium Triangle*; Such are the *First* and *Second Triangles A B C*, and *B D G.*

Note, *All Triangles that have not a Right Angle, whether they are Acute, or Obtuse, are in General Terms, called Oblique Triangles,*

es, without any other Diſtinction, as before. And the longeſt de of every Oblique Triangle is uſually called the Baſe; *the her two are only called* Sides *or* Legs.

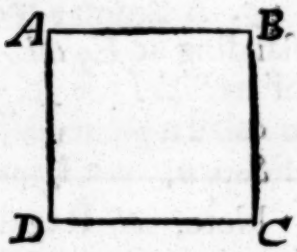
8. The **Altitude** or **Height** of any *lain Triangle*, is the *Length* of a *Right ine* let fall *Perpendicular* from any of its *ngles*, upon the *Side* oppoſite to that *ngle* from whence it falls; and may be *ther* within, or without the *Triangle*, *Occaſion requires*, being denoted by *e Two prick'd Lines*, in the *annexed riangles*.



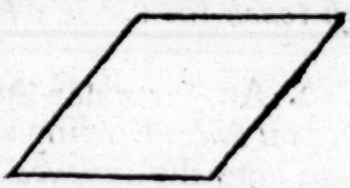
Sect. 4. Of Four-ſided Figures, &c.

1. A **Square** is a plain *regular Figure*, *hoſe Area* is *limited* by *Four* equal *Sides* *Perpendicular* one to another.

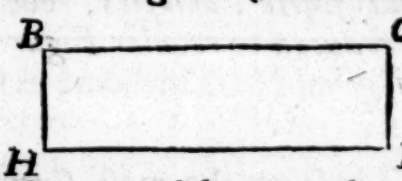
That is, when $AB = BC = CD = DA$, *d* the *Angles A, B, C, D* are all *equal*, *hen* it's uſually called a *Geometrical quare*.



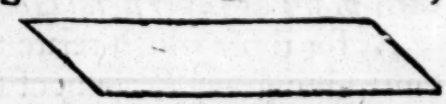
2. A **Rhombus**, or *Diamond-like igure*, is that which hath *Four* equal *des*, but no *Right Angle*. That is, *Rhombus* is a *Square* mov'd out of its *ght Poſition*, as the *annexed Figure*.



3. A **Rectangle**, or a *Right-angled Parallelogram* (often called *Oblong*, or *long Square*) is a *igure* that hath *four Right Angles* *d* its *two Oppoſite Sides* equal, *z.* $BC = HD$ and $BH = CD$.



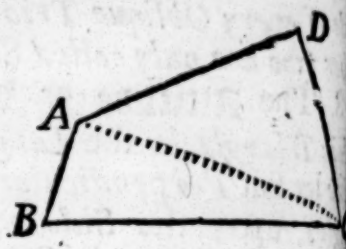
4. A **Rhomboides**, is an *Oblique-angled Parallelogram*; that is, *is a Parallelogram Moved* out of *Right Poſition*, like the *An-xed Figure*.



5. The **Altitude** or *Height* of any *Oblique-angled Parallelogram*, *z.* either of the *Rhombus* or *Rhomboi-s*, is a *Right-Line* let fall *perpendicu-* from any *Angle* upon the *Side oppo-e* to that *Angle*; and may either be *ithin* or without the *Figure*: As the *ick'd Lines* in the *annexed Figure*.

6. All *Four-ſided Figures*, which *differ* from thoſe before-mention'd, are called **Trapezia**.

That is, when they have neither *Oppoſite Sides*, nor *Oppoſite Angles*, *Equal*; as the *Figure A B C D*



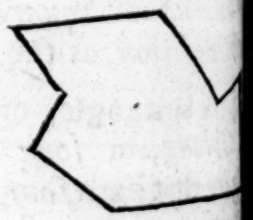
7. A *Right-line* drawn from any *Angle* in a *four-ſided Figu* to its *oppoſite Angle*, is call'd a **Diagonal** *Line*, and will *Divi* the *Area* of the *Figure* into *two Triangles*, being *denoted* by t *prick'd Line A C* in the laſt *Figure*.

8. All *right-Sined Figures* that have more than *four Sid* are call'd *Polygons*, whether they be *regular* or *irregular*.

9. A **Regular Polygon** is that which hath all its *Sides Equ* ſtanding at *Equal Angles*, and is named according to the *Numb* of its *Sides* (or *Angles*). That is, if it have *Five Equal Sides*, is call'd a **Pantagon**; if *Six Equal Sides*, it is call'd a **Hexago** if *Seven*, 'tis a **Heptagon**; if *Eight*, 'tis an **Octagon**, &c.

Note, *All* Regular Polygons *may be inſcrib'd in a Circle*: th *is, their Angular Points, how many ſoever they have, will just touch the Circle's Periphery.*

10. An **Irregular Polygon** is that *Figure* which hath ma *unequal Sides* ſtanding at *unequal Angles* (like unto the *annexed Figure*, or otherwiſe); and of ſuch kind of *Polygons* there are *infinite Varieties*, but they may all be *reduced* to *regular Figures* by drawing *Diagonal Lines* in them; as ſhall be ſhew'd farther on.



Theſe are the moſt *General* and *Uſeful Definitions* that co cern *plain* or *ſuperficial Geometry*.

As for thoſe which relate to *Solids*, I thought it convenient omit giving any Account of them in this place, becauſe they wo rather *puzzle* and *amuſe* the *Learner*, than *improve* him, until h has gain'd a *competent Knowledge* in the moſt uſeful *Theore* concerning *Superficies*, for then thoſe *Definitions* may be m eaſily underſtood, and will help to form a *clearer Idea* of th *reſpective Solids*, than 'tis *poſſible* to conceive of them befor and therefore I have reſerv'd thoſe *Definitions* until we come the Fifth Part.

Sec

ect. 5. *Of such* Terms *as are generally used in Geometry.*

Whatsoever is proposed in *Geometry* will either be a Problem a Theorem.

Both which *Euclid includes* in the *general Terms* of *Proposition.*

A Problem is that which *proposes* something to be *Done*, and ates more immediately to *practical* than *speculative Geometry*; hat is, it's generally of such a *Nature*, as to be perform'd by ne known or *common-receiv'd* Rules, without any Regard had their *Inventions* or *Demonstrations.*

A Theorem is when any *Common-receiv'd* Rule, or any *New* oposition is requir'd to be *Demonstrated*, that so it may from nceforward become a *certain* Rule, to be *rely'd* upon in Pra- e when Occasion requires it. And therefore several Rules are en call'd *Theorems*, by which *Operations* in *Arithmetick*, and nclusions in *Geometry*, are *perform'd.*

Note, *By* Demonstration *is understood the highest Degree of* oof *that Human Reason is capable of attaining to, by a Train of* guments *deduc'd or drawn from such plain Axioms, and other* f-evident *Truths, as cannot be denied by any One that consi-* rs *them.*

A Corollary, or Consectary, is some *Consequent Truth* drawn or n'd from any *Demonstration.*

A Lemma is the *Demonstration* of some *Premises* laid down or oposed as *preparative* to obviate and shorten the Proof of the eorem under Consideration.

A Scholium is a brief *Commentary* or *Observation* made upon ne *precedent Discourse.*

N.B. *I advise the Young Geometer to be very perfect in the* finitions, *viz. Not to rest satisfied with a bare Remembrance them; but, that he endeavour to gain a clear* Idea *or* Under- nding *of the Things defined; and for that reason I have been ller in every* Definition *than is usual.*

And, that he may know from whence most of the following oblems *and* Theorems *contain'd in the Two next Chapters are lected, I have all-along cited the* Proposition, *and* Book *of* clid's Elements *where they may be found.*

As for instance; at Problem 1. *there is* (3. *e.* 1) *which shews t it is the* Third Proposition *in* Euclid's First Book. *The like st be understood in the* Theorems.

CHAP. II.

The First Rudiments, or *Leading* and *Preparatory* Problems in Plain Geometry.

In order to perform the following Problems, *the young Geometer ought to be provided with a thin ſtreight* Ruler, *made either of* Braſs *or* Box-wood, *and two pair of very good* Compaſſes, *viz. one pair call'd* Three-pointed Compaſſes, *being very uſeful for drawing of* Figures *or* Schemes, *either with black-Lead or Ink, and one pair of* plain Compaſſes *with very fine Points, to meaſure and ſet off Diſtances; alſo he ſhould have a very good Steel* Drawing-Pen: *And then he may proceed to the Work with this Caution; That he ought to make himſelf Maſter of one* Problem *before he nndertake the next: That is, he ought to underſtand the* Deſign, *and, as far as he can, the Reaſon of every* Problem *as well as how to do it; and then a little Practice will render them very Eaſy, they being all grounded upon theſe following* Poſtulates.

Poſtulates *or* Petitions.

1. That a *Right-line* may be *Drawn* from any one given *Point* to another.

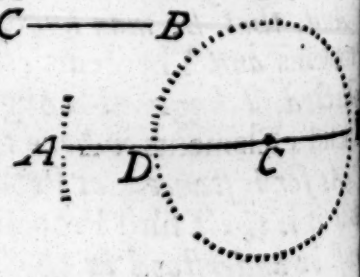
2. That a *Right-line* may be *produced*, *encreaſed*, or made *longer* from either of its Ends.

3. That upon any given *Point* (or *Centre*) and with any given *Diſtance* (*viz.* with any Radius) a *Circle* may be deſcribed.

PROBLEM I.

Two Right-lines being given; To find their Sum and Difference. (3. *e.* 1.)

Let the given *Lines* be $\begin{cases} A \text{——} C \\ C \text{——} B \end{cases}$



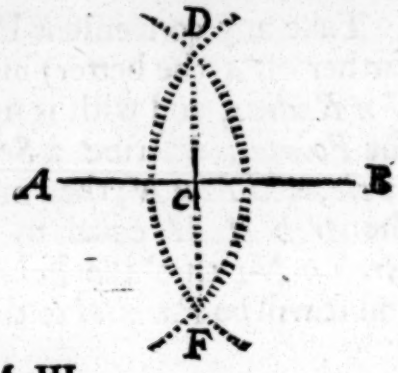
Make the *ſhorteſt Line*, as *C B*, *Radius*, and with it *Deſcribe* a *Circle*: From its Centre *C* ſet off the other *Line A C*, and *joyn A C B* with a *Right-line*. Then will $AB = AC + CB$; and $AD = AC - CB$; as was required.

PRO

PROBLEM II.

To Biſect or Divide a Right-line given (as A B) into two Equal Parts (10. *e.* 1.)

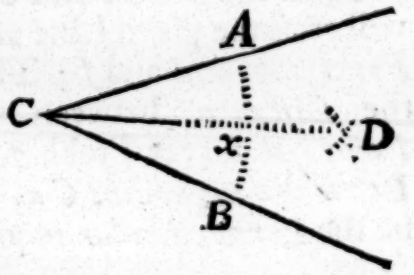
From both *Ends* of the given *Line* (*viz. A* and *B*) with any *Radius Greater* than *half* its *length*, *deſcribe Two Arches* that may *croſs* each other in *Two Points*, as at *D* and *F*; then *joyn* thoſe *Points D F* with a *Right-line*, and it will *biſect* the *Line A B* in the middle at *C*; *viz.* it will make *A C* = *C B*; as was required.



PROBLEM III.

To Biſect a Right-lin'd Angle given, into Two equal Angles (9. *e.* 1.)

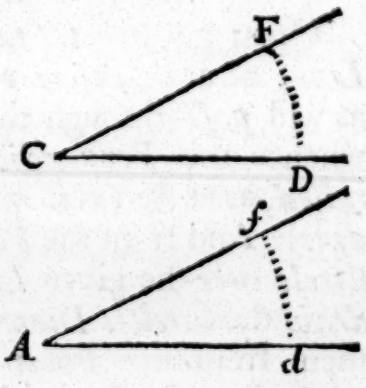
Upon the *Angular Point*, as at *C*, with any *convenient Radius*, *deſcribe* an *Arch* as *A B*; and from thoſe *Points A* and *B*, *Deſcribe two* equal *Arches croſſing* each other, as at *D*; then *joyn* the *Points C* and *D* with a *Right-line*, and it will *biſect* the *Arch A B*, and conſequently the *Angle*; as was requir'd.



PROBLEM IV.

At a Point A, in a Right-line given A B; to make a Right-lin'd Angle equal to a Right-lin'd Angle given C. (23. *e.* 1.)

Upon the given *Angle Point C* deſcribe an *Arch*, as *F D*, (making *C D* any *Radius* at *pleaſure*) and with the ſame *Radius deſcribe* the like *Arch* upon the given *Point A*, as *f d*; that is, make the *Arch f d* Equal to the *Arch F D*; Then *joyn* the *Points A* and *f* with a *Right-line*, and it will form the Angle requir'd.

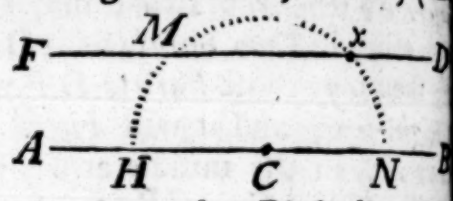


PRO-

PROBLEM V.

To draw a Right-line, as F D, parallel to a given Right-line A B that ſhall paſs thro' any aſſign'd Point, as at x, viz. *at any Diſtance requir'd.* (31. *e.* 1.)

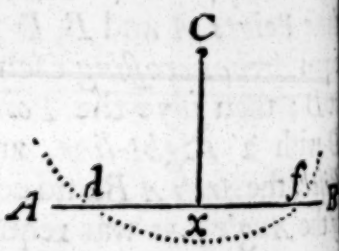
Take any convenient Point in the given *Line*, as at *C*, (the farther off *x* the better) make *C x Radius*, and with it upon the *Point C*, deſcribe a *Semi-circle*, as *H M x N*; then make the *Arch H M* equal to the *Arch x N*; thro' the *Points M* and *x* draw the *Right-line F D*, and it will be *Parallel* to the *Line A C*, as was requir'd.



PROBLEM VI.

To Let fall a Perpendicular, *as G x, upon a given Right-line A B from any aſſign'd Point that is not in it, as from C.* (12. *e.* 1)

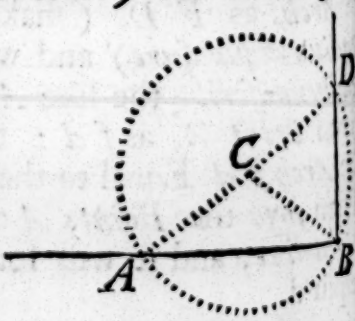
Upon the given Point *C*, *deſcribe* ſuch an *Arch* of a *Circle* as will croſs the given Line *A B* in *two Points*, as at *d* and *f*; Then *Biſect* the *Diſtance* between thoſe *two Points d f* (per Probl. 2.) as at *x*. *Draw* the *Right-line C x*, and it will be the *Perpendicular* requir'd.



PROBLEM VII.

To Erect or Raiſe a Perpendicular *upon the End of any given Right-line, as at B; or upon any other Point aſſign'd in it.* (11. *e.* 1.)

Upon any *Point (taken at an Adventure)* out of the given *Line*, as at *C*, *deſcribe* ſuch a *Circle* as will *paſs* through the *Point* from whence the *Perpendicular* muſt be *raiſed*, as at *B*, (viz. *make C B Radius*): And from the *Point* where the *Circle cuts* the given *Line*, as at *A*, *draw* the *Circle's Diameter A C D*; then from the *Point D* draw the *Right-line D B*, and it will be the *Perpendicular* as was required.

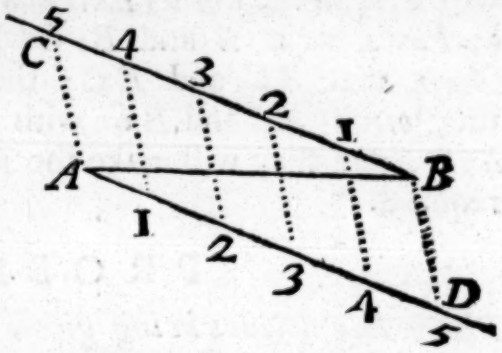


PRO

PROBLEM VIII.

To Divide any given Right-line, as A B, into any propoſed Number of Equal parts. (10. *e.* 6.)

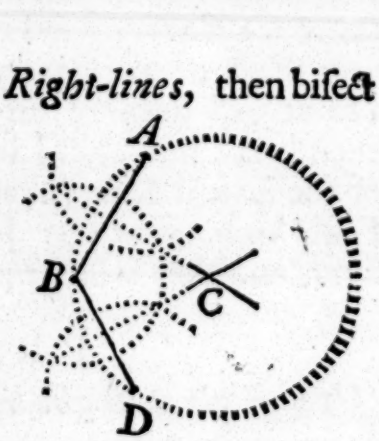
At the Extream Points (or Ends) of the given Line, as at *A* and *B*, make *Two* equal *Angles* (by *Prob.* 4.) continuing their Sides *A D* and *B C* to any ſufficient *length*; then upon thoſe Sides, beginning at the Points *A* and *B*, ſet off the propoſed *Number* of Equal Parts (*ſuppoſe 'em* 5.) If *Right-lines* be drawn (*croſs the given Line*) from one Point to the other, as in the annex'd Figure, thoſe *Lines* will divide the given *Line A B* into the *Number* of equal Parts requir'd.



PROBLEM IX.

To Deſcribe a Circle that ſhall paſs (or cut) thro' any Three Points given, not lying in a Right-line, as at the Points A B D.

Joyn the Points *B A* and *B D* with *Right-lines*, then biſect both thoſe *Lines*, (*per Problem* 2.) the Point where the biſecting Lines meet, as at *C*, will be the *Centre* of the *Circle* requir'd.



The Work of this Problem being well underſtood, 'twill be *eaſie* to perform the two following, without any *Scheme*, viz.

1. *To find the Centre of any Circle given.* (1. *e.* 3.)

By the laſt Problem 'tis plain, that if *three* Points be any where taken in the given *Circle's Periphery*, as at *A*, *B*, *D*, the *Centre* of that Circle may be found as before.

2. *If a Segment of any Circle be given, to compleat or Deſcribe the whole Circle.*

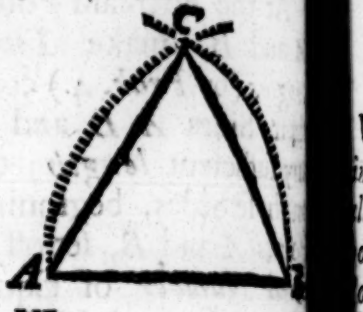
This may be done by taking any *Three Points* in the given *Segment's* Arch, and then proceed as before.

PRO-

PROBLEM X.

Upon a Right-line given, as A B; To describe an Equilateral Triangle. (1. *e.* 1.)

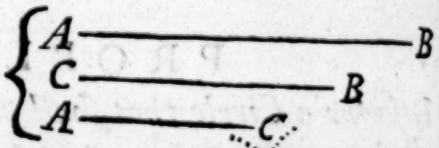
Make the given Line *Radius*, and with it, upon each of its *Extream Points* or *Ends*, as at *A* and *B*, describe an *Arch*, *viz.* *A C* and *B C*; then joyn the *Points A C* and *B C* with *Right-lines*, and they will make the *Triangle* requir'd.



PROBLEM XI.

Three Right-lines being given; To form them into a Triangle (provided any Two of them, taken together, be longer than the Third.) (22. *e.* 1.)

Let the given *Lines* be

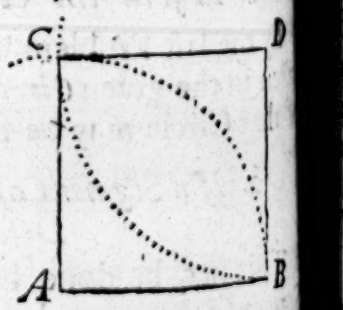


Make either of the *ſhorter Lines* (as *A C*) *Radius*, and upon either End of the *longeſt Line* (as at *A*) deſcribe an Arch; then make the other Line *C B Radius*, and upon the other End of the *longeſt Side* (as at *B*) deſcribe another Arch, to croſs the Firſt Arch (as at *C*): Joyn the *Points C A* and *C B* with *Right-lines*, and they will form the *Triangle* requir'd.

PROBLEM XII.

Upon a given Right-line, as A B, to form a Square. (46. *e.* 1.)

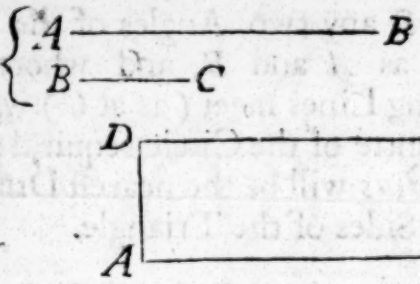
Upon one End of the given *Line*, as at *B*, erect the *Perpendicular B D*, equal in Length with the given *Line*, *viz.* make *B D* = *A B*; that being done, make the given Line *Radius*, and upon the *Points A* and *D* deſcribe equal Arches to croſs each other, as at *C*; then joyn the *Points C A* and *C D* with *Right-lines*, and they will form the *Square* requir'd.



PRO-

PROBLEM XIII.

wo Unequal Right-lines be ing given ; To form or make of them a Right-angled Parallelogram.



Let the given *Lines* be

Upon one End of the *Longeſt ine*, as at *B*, erect a *Perpendi-lar* of the ſame length with the *orteſt Line B C*; then from the-*oint C* draw a Line parallel, and the ſame length, to *A B*, *viz.* make *D C = A B:* Joyn *D A* ith a *Right-line*, and it will form the *Oblong* or *Parallelogram* quir'd

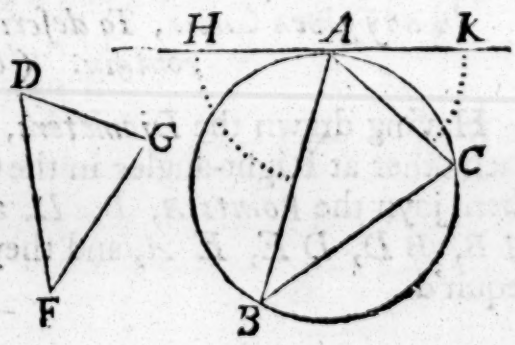
As for *Rhombus's* and *Rhomboides*, to wit, *Oblique-angled* arallelograms, they are made, or deſcrib'd, after the ſame man-r with the two laſt Figures ; only inſtead of erecting the Per-ndiculars, you muſt ſet off their given Angles, and then proceed draw their Sides parallel, *&c.* as before.

PROBLEM XIV.

any given Circle, To inſcribe or make a Triangle, whoſe An-gles ſhall be Equal to the Angles of a given Triangle; as the Triangle F D G. (2. *e.* 4.)

Note, *Any Right-lined Figure is ſaid to be inſcrib'd in a Cir-when all the Angular Points of that Figure do juſt touch e Circle's Periphery.*

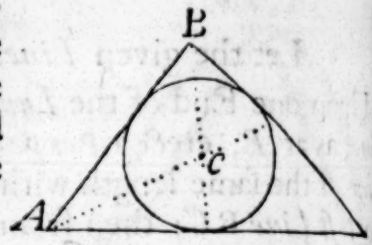
Draw any *Right-line* (as *H K*) ſo as juſt to touch the Circle, at *A*; then make the Angle *A C* equal to any one Angle the given Triangle, as *D F G*; d the Angle *H A B* equal to other Angle of the Triangle, *D G F*; then will the Angle *A C* be equal to the Angle *D G*. Joyn the *Points B* and with a Right-line, and 'twill rm the Triangle requir'd.



PROBLEM XV.

In any given Triangle, as A B D, To describe a Circle that shall touch all its Sides. (4. *e.* 4.)

Bisect any two Angles of the Triangle, as *A* and *B*, and where the bisecting Lines meet (as at *C*) will be the Centre of the Circle requir'd; and its *Radius* will be the nearest Distance to the Sides of the Triangle.



PROBLEM XVI.

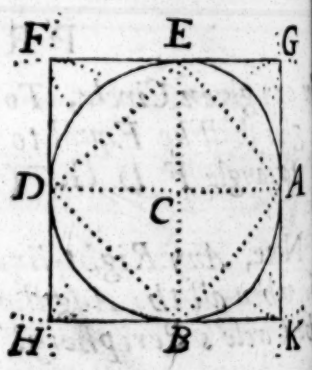
To describe a Circle about any given Triangle. (5. *e.* 4.)

This *Problem* is perform'd in all respects like the Ninth, *viz.* by bisecting any *Two Sides* of the given Triangle; the Point where those bisecting Lines meet, will be the Centre of the Circle requir'd.

PROBLEM XVII.

To describe a Square about any given Circle. (7. *e.* 4.)

Draw two *Diameters* in the given Circle (as *D A* and *E B*) ar Right-Angles in the Centre *C*; and with the Circle's *Radius C A* describe from the *Extream Points* of the Diameter *A B*, *D E*, cross Arches, as at *F G H K*; then joyn those *Points* where the Arches cross with *Right-lines*, and they will form the *Square* requir'd.



PROBLEM XVIII.

In any given Circle, To describe the largest Square it can contain. (6. *e.* 4.)

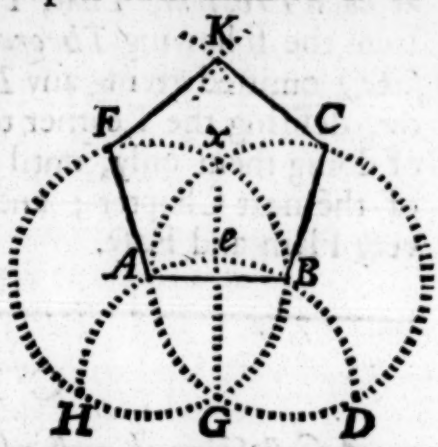
Having drawn the *Diameters*, as *D A* and *E B*, bisecting each other at Right-angles in the Centre *C*, (as in the last Scheme) then joyn the *Points A*, *B*, *D*, and *E*, with *Right-lines*, viz. *A B*, *B D*, *D E*, *E A*, and they will be the Sides of the *Square* requir'd.

PRO-

PROBLEM XIX.

Upon any given Right-line, as A B, To describe a Regular Pentagon, *or Five-sided* Polygon.

Make the given Line *Radius*, and upon each End of it describe a Circle; and through those Points where the Circles cross each other (as at *G x*) draw the Right-line *G e x*, upon the Point *G*; with the same *Radius* describe the Arch *H A e B D*: Then lay a Ruler upon the Points *D e*, and mark where it *crosses* the other Circle, as at *F*. Again, lay the Ruler upon the Points *H e*, and mark where it *crosses* the other Circle, as at *C*: Then from the Points *F* and *C* *(with the same Radius as before)* describe cross Arches, as at *K*: Joyn the Point *A F*, *F K*, *K C*, and *C B*, with Right-lines, and they will form the *Pentagon* requir'd, *viz.* *A F* = *F K* = *K C* = *C B* = *A B*; and the Angles at *A*, *B*, *C*, *K*, *F* will be Equal.

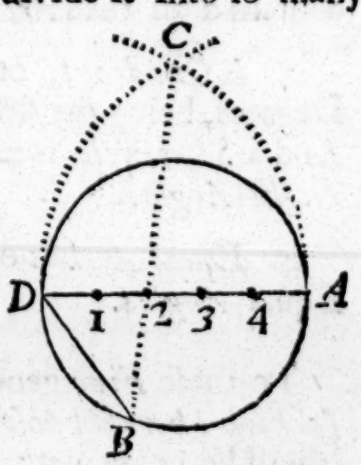


PROBLEM XX.

In any given Circle, To describe a Regular Pentagon. (11. *e.* 4. & 10. *e.* 3.)

Or, in General Terms, to describe any Regular Polygon *in a Circle.*

Draw the Circle's Diameter *D A*, and divide it into so many equal parts as the proposed *Polygon* hath number of Sides; then make the whole Diameter a *Radius*, and describe the two Arches *C A* and *C D*. If a Right-line be drawn from the Point *C*, thro' the Second of those equal parts in the Diameter, as at 2, it will assign a Point in the opposite Semicircle's *Periphery*, as at *B*. Joyn *D B* with a Right-line, and it will be the true Side of the *Pentagon* requir'd.



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CHAP. III.

A Collection of most useful Theorems *in plain Geometry* Demonstrated.

Note, *In order to shorten several of the following Demonstrations, it will be necessary to premise, That*

1. THE *Periphery* (or Circumference) of every Circle (*whether Great or Small*) is suppos'd to be divided into 360 equal parts, called *Degrees*; and every one of those *Degrees* are divided into 60 *equal parts*, call'd *Minutes*, &c.

2. All *Angles* are measur'd by the Arch of a Circle describ'd upon the *Angular Point* (*see Defin. 9, page* 287.) and are esteem'd *Greater* or *Less*, according to the Number of Degrees contain'd in that *Arch*.

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4. *Equal Arches* of a *Circle*, or of Equal Circles, *Measure* Equal *Angles*.

To those *Five* general *Axioms* already laid down in *page* 146 *(which I here suppose the Reader to be very well acquainted with)* it will be convenient to understand these following, which begin their *Number* where the other ended.

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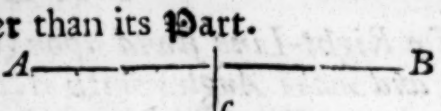
That is, the whole Line *A B* is *greater* than its *Part A c*, &c.



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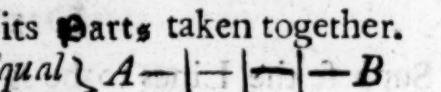

The ſame is alſo true in *Superficies* and *Solids*.

8. Thoſe *Things* which, being laid one upon another, do *agree* or *meet* in all their Parts, are Equal one to the other.

But the Converſe of this *Axiom*, to wit, that *Equal Things* being laid one upon the other will meet, is only *true* in *Lines* and *Angles*, but not in *Superficies*, unleſs they be alike, *viz.* of the ſame Figure or Form: As for inſtance, a Circle may be *Equal* in *Area* to a Square, but if they are *Laid* one upon the other, 'tis plain they cannot *meet* in all their *parts*, becauſe they are *unlike Figures*. Alſo a *Parallelogram* and a *Triangle* may be Equal in their *Area's* one to another, and both of them may be Equal to a *Square*, but if they are *Laid* one upon the other, they will not *meet* in all their *Parts*, &c.

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THEO-

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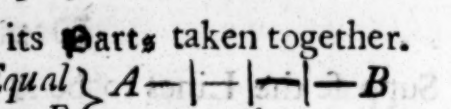

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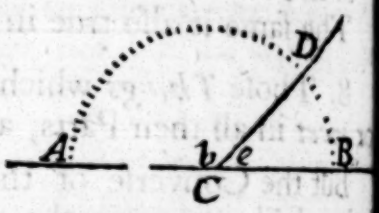
THEO-

THEOREM I.

If a Right-Line ſtand upon (or meet with) another Right-Line and make Angles with it, they will either be Two Right-Angles, or Two Angles equal to Two Right-Angles. (13. *e.* 1.)

Demonſtration.

Suppoſe the Lines to be *A B* and *D C*, meeting in the Point at *C*; upon *C* deſcribe any Circle at pleaſure: Then will the Arch *A D* be the *Meaſure* of the $\prec b$, and the Arch *D B* the *Meaſure* of the $\prec e$, but the Arches *A D* + *D B* = 180°, *viz.* they compleat the *Semicircle*.



Conſequently the $\prec b + \prec e = 180°$, *which was to be prov'd.*

Corollaries.

1. Hence it follows, that if the $\prec b = 90°$ then $\prec e = 90°$ but if $\prec b$ be *Obtuſe*, then the $\prec e$ will be *Acute*, &c.

From hence it will be eaſie to conceive, that if ſeveral Right Lines ſtand upon, or meet with, any Right-Line at one and the ſame Point, all the Angles taken together will be = 180°, *viz.* Two Right-Angles.

THEOREM II.

If Two Angles interſect (viz. *cut or croſs*) *each other, the Two Oppoſite Angles will be Equal.* (15. *e.* 1.)

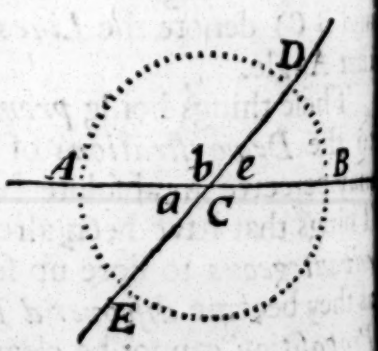
Demonſtration.

Let the *Two Lines* be *A B* and *D E*, interſecting each other in the Centre *C*.

Then $\prec b + \prec e = 180°$
And $\prec b + \prec a = 180°$ } *per* laſt

Conſequently $\prec b + \prec e = \prec b + \prec a$, *per Axiom* 5.

Subſtract $\prec b$ on both Sides of the *Æquation*, and it will leave $\prec e = \prec a$.



Again, $\prec b + \prec e = 180°$, as before; and $\prec e + \prec C = 180°$ conſequently $\prec e + \prec C = \prec b + \prec e$. *Subſtract* $\prec e$, and then $\prec C = \prec b$. Q. E. D.

Coro

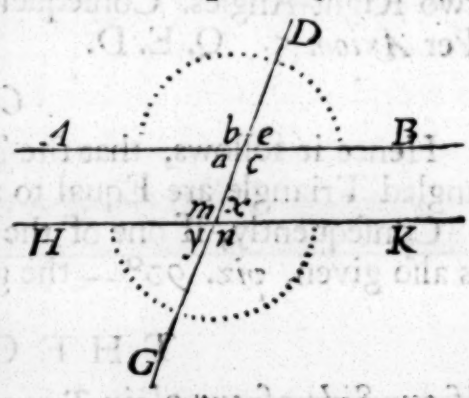
Corollary.

From hence it is evident, that if Two Lines interſect each ther, they will make Four Angles ; which being taken together, ill always be Equal to Four Right-Angles.

THEOREM III.

f a Right-Line cut (or croſs) Two Parallel Lines, it will make the Oppoſite Angles equal one to another. (29. *e.* 1.)

Suppoſe the Two Lines *A B* and *H K* to be parallel, and the Right-line *D G* to cut them both t *C* and *n*: Upon the Point *C* (with *ny Radius*) deſcribe the Semicir-le; and with the ſame *Radius*, up-n the Point at *n*, deſcribe another emicircle oppoſite to the firſt, as in he Figure. Then 'tis plain, and I uppoſe very eaſy to conceive, that f the Centre *C* were mov'd along pon the Line *D G*, until it came o the Centre at *n*, the two Lines *A B* and *H K* would *meet* nd concur, *viz.* become one Line (*for Parellel Lines are as it were but one broad Line*). Conſequently the Two Semicircles ould alſo meet, and become one entire Circle, like to that in the aſt *Demonſtration.*



nd therefore the $\angle y = \angle x = \angle a = \angle e$ } { as before, *per*
And $\angle m = \angle n = \angle b = \angle c$ } { *laſt Theorem.*

Q. E. D.

Corollary.

Hence it follows, that if *Three*, *Four*, or never ſo many Paral-el-lines, are cut or croſs'd by one Right-line, all their Oppoſite ngles will be Equal.

THEOREM IV.

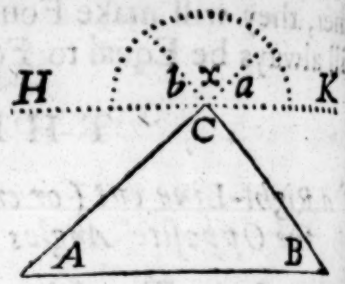
he Three Angles of every plain Triangle, are Equal to Two Right-Angles; (32. *e.* 1.)

onſequently, any Two Angles of any plain Triangle muſt needs be Leſs than Two Right-Angles. (17. *e.* 1.)

Demon-

Demonſtration.

Let the △ *A B C* be propos'd; draw the Right-line *H K* parallel to the Side *A B*, juſt touching the Vertical Angle *C*; And upon the ſame Angular Point *C* deſcribe any Semicircle, and produce the Sides *A C*, and *B C* to its *Periphery*. Then will ≺ *b* = ≺ *B*, ≺ *a* = ≺ *A*, and ≺ *x* = ≺ *C*, *per* Laſt *Theorem*.



But ≺ *b* + ≺ *a* + ≺ *x* = 180°, or two Right-Angles. Conſequently ≺ *B* + ≺ *A* + ≺ *C* = 180°. Per *Axiom* 5. Q. E. D.

Corollary.

Hence it follows, that the Two *Acute* Angles of every Right-angled Triangle are Equal to a Right-Angle, or 90°.

Conſequently, if one of the *Acute* Angles be given, the other is alſo given, *viz.* 90° — the given ≺ leaves the other ≺

THEOREM V.

If one Side of any plain Triangle be continued or produced beyond, or out of the Triangle, the outward Angle will always be Equal to the Two inward Oppoſite Angles. (32. *e.* 1.)

Demonſtration.

Let the Side *A B* of the △ *A B C* be produced out of the △, ſuppoſe to *D*, &c. as in the Figure.

Then ≺ *z* = ≺ *A* + ≺ *C*, for the ≺ *B* + ≺ *z* = 180° per *Theorem* 1.

And the ≺ *B* + ≺ *A* + ≺ *C* = 180°, *per* laſt *Theorem*.

Therefore ≺ *B* + ≺ *z* = ≺ *B* + ≺ *A* + ≺ *C*, per Axiom 5. *Subſtract* ≺ *B* on both Sides the *Æquation*, and it will leave ≺ *z* = ≺ *A* + ≺ *C* (*per Axiom* 2.) Q. E. D.

Conſequently, the outward Angle (at z) of any plain Triangle muſt needs be greater than either of the inward oppoſite Angles, viz. *greater than* ≺ *A, or* ≺ *c* (16. *e.* 1.)

Corollary.

Hence it follows, that if One Angle of any plain Triangle be given, the *Sum* of the other Two Angles is alſo given for 180° — the given ≺ = the other Two ≺ ≺.

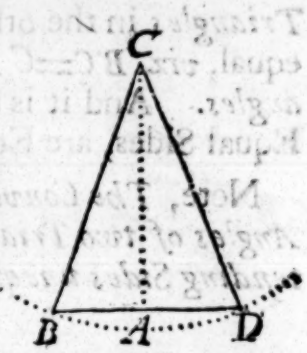
THEO-

THEOREM VI.

In every plain Triangle, Equal Sides subtend (viz. *are opposite to*) *Equal Angles.* (5. *e.* 1.)

Consequently, Equal Angles are subtended by Sides. (6. *e.* 1)

Demonstration.



Suppose the $\triangle BCD$ to be an *Isosceles* $\triangle$; that is, let $BC = CD$. Bisect the $\angle C$, or (*which is all one*) make CA *Perpendicular* to BD; then will the $\angle\angle$ on each side of it (*viz.* $\angle BAC$ and $\angle DAC$) be Right Angles.

Therefore $\left\{\begin{matrix} \frac{1}{2}\angle C + \angle B = 90^\circ \\ \frac{1}{2}\angle C + \angle D = 90^\circ \end{matrix}\right\}$ Per *Corol.* to *Theorem* 4.

Consequently, $\frac{1}{2}\angle C + \angle B = \frac{1}{2}\angle C + \angle D$, per *Axiom* 5. Subtract $\frac{1}{2}\angle C$ from both sides of the *Æquation*, and it will have $\angle B = \angle D$, per *Axiom* 2. Q. E. D.

Corollary.

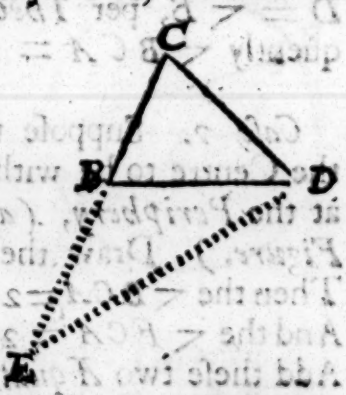
From hence it follows, that the *Three Angles* of an *Equilateral Triangle* are Equal one to another.

THEOREM VII.

In every plain Triangle, the Longest Side subtends the Greatest Angle. (18. *e.* 1.)

Consequently, the Greatest Angle of any plain Triangle is subtended by the Longest Side.

This *Theorem* is evident by Inspection only: For, let one of the Sides of any *plain Triangle* (as CB) be produced, suppose to E; joyn DE with a *Right-Line*; Then 'tis evident, that because CE is now made longer than the Side BC, therefore the $\angle$ at D is become *larger* than it was before by the $\angle BDE$: And it's plain, the longer the Side CE had been made, the $\angle$ at D would have been the more enlarg'd.



THEOREM VIII.

If the Sides of Two Triangles are Equal, the Angles oppoſite to thoſe Equal Sides will be Equal. (8. *e.* 1.)

The Truth of this *Theorem* is evident by the Two included *Triangles* in the 6th *Theorem*, for they have their reſpective Sides equal, *viz.* $BC = CD$, $BA = DA$, and CA common to both *Triangles*. And it is there prov'd, That the $\angle$ oppoſite to the Equal Sides, are Equal, &c. which needs no further *Proof*.

Note, *The Converſe of this* Theorem *holds not true; for the Angles of two Triangles may be equal, and their oppoſite or ſubtending Sides unequal; as will appear at* Theorem XII.

Corollary.

Hence it follows, that *Triangles mutually Equilateral* are alſo *mutually Equiangular*; and,

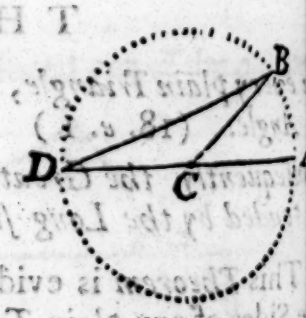
That *Triangles mutually Equilateral* are Equal one to another. (4. & 26. *e.* 1.)

THEOREM IX.

An Angle at the Centre of any Circle, is always double to the Angle at the Periphery, when both the Angles ſtand upon the ſame Arch. (20. *e.* 3.) This *Theorem* hath Three Varieties or Caſes.

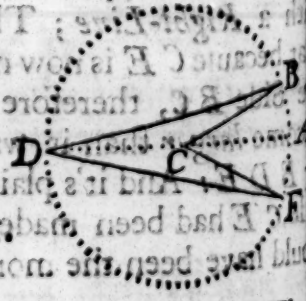
Demonſtration.

Caſe 1. Let the Diameter DA, and the Line DB, be the two Lines which form the $\angle$ at the *Periphery*; Draw the *Radius* BC, then $\angle BCA$ the $\angle$ at the *Centre*.



But $\angle BCA = \angle D + \angle B$, per *Th.* 5.
And becauſe $DC = BC$, therefore $\angle D = \angle B$, per *Theorem* 6. Conſequently $\angle BCA = 2 \angle D$.

Caſe 2. Suppoſe the $\angle BCF$ at the Centre to be within the $\angle BDF$ at the *Periphery*, (*as in the annexed Figure.*) Draw the Diameter DA;
Then the $\angle BCA = 2 \angle BDA$
And the $\angle FCA = 2 \angle FDA$ } per *Caſe* 1.
Add theſe two *Æquations* together:



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Then will $\angle BCA + \angle FCA = 2\angle BDA + 2\angle FDA$, per *Ax.* 1.
But $\angle BCA + \angle FCA - \angle BCF$
And $2\angle BDA + 2\angle FDA = 2\angle BDF$
Consequently $\angle BFC = 2\angle BDF$.

Case 3. Again, suppose the $\angle BCF$ at the Centre to be out of the $\angle BDF$ at the Periphery: From the Angular Point D at the Periphery draw the Diameter DA.

Then $\angle FCA = 2\angle FDA$
And $\angle BCA = 2\angle BDA$ } per Case 1

Subtract this last *Æquation* from the other, and it will leave
$\angle FCA - \angle BCA = 2\angle FDA - 2\angle BDA$, per *Axiom* 2.
But $\angle FCA - \angle BCA = \angle FCB$. And $2FDA - 2\angle BDA = 2\angle FDB$. Consequently, $\angle FCB = 2\angle FDB$. Q.E.D.

Corollary.

Hence 'tis evident, That all Angles at the Periphery, which stand on the same *Segment* or *Arch* of a Circle, or upon equal Arches, are equal one to another. (21. *e.* 3.)

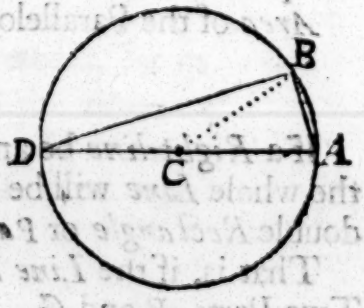
THEOREM X.

An Angle in a Semicircle is a Right-angle. (31. *e.* 3.)

That is, *If the Diameter of any* Circle *be the Side of a Triangle, and the Angle opposite to that Side be any where in the* Circle's *Periphery, it will be a Right-angle.*

Demonstration.

Let DA be the Diameter, and DBA the Triangle, then $\angle B = 90°$. Draw the Radius BC, then is the $\angle DBA = \angle D + \angle A$.



For $\angle CBD = \angle D$, and $\angle CBA = \angle A$, per *Theorem* 6.
Therefore $\angle DBA = \angle CBD + \angle CBA$, per *Axiom* 5.
Again, $\angle DBA + \angle D + \angle A = 180°$, per *Theorem* 4.
Consequently, $\angle DBA = 90°$ or a Right-angle. Q. E. D.

Corollaries.

1. Hence it will be eaſy to conceive, that an Angle made in any Segment leſs than a Semicircle will be *Obtuſe*, or greater than a Right-angle.

2. And an Angle made in any Segment greater than a Semicircle, muſt conſequently be *Acute*.

THEOREM XI.

In any Right-angled Triangle, the Square which is made of the Hypothenuſe, or Side ſubtending the Right-angle, is equal to both the Squares which are made of the Sides including the Right-angle. (47. *e.* 1.)

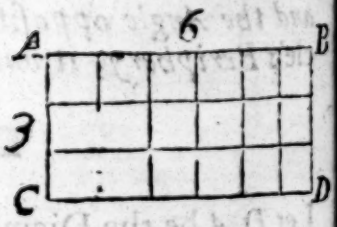
There are ſeveral ways of demonſtrating this Noble and Uſeful *Theorem*, but, I preſume, none more eaſy to be underſtood by a *Learner* than that which I ſhall here propoſe: And, in order thereto, 'twill be neceſſary to premiſe the following *Lemma's*.

Lemma 1.

A *Right-line* is ſaid to be *Multiply'd* with a *Right-line* when either a *Square*, or other *Right-angled Parallelogram*, is made of the *Two Lines*.

That is, the *Area* of any *Right-angled Parallelogram* is equal to the *Product* of thoſe *Numbers* which expreſs the *Meaſure* of its Sides.

Thus, if $AB = 6$ Inches,
And $AC = 3$ Inches;
Then $AB \times AC = 6 \times 3 = 18$ Square Inches; which is the *Area* of the Parallelogram $ABCD$.



Lemma 2.

If a *Right-line* be any way cut into *two parts*, the *Square* of the whole *Line* will be Equal to the *Squares* of each Part, and a double *Rectangle* or *Parallelogram* made of both the Parts, (4. *e.* 2.)

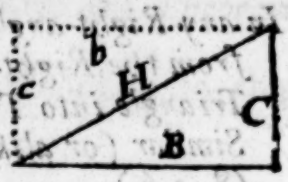
That is, if the *Line S* be cut into the Two Parts *B* and *C*,
Then is $S = B + C$; but if both the Sides of the *Æquation* be involv'd, it will be $SS = BB + 2BC + CC$.

S
B | C

Lemma

Lemma 3.

The *Area* of every *Right-angled Triangle* is half the *Parallelogram* made of its Base and Perpendicular.



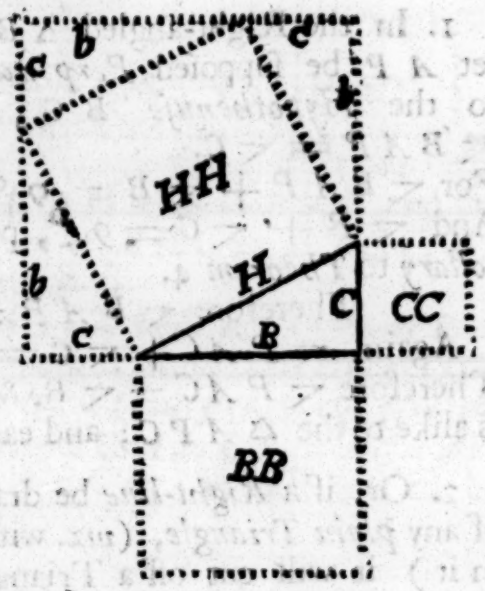
For BC = the *Area* of the whole *Parallelogram*, by the first *Lemma*. And $\triangle BCH + \triangle bcH$ = the *Parallelogram*; but $B = b$, and $C = c$.

Therefore $\frac{1}{2}BC$ = the *Area* of each $\triangle$, *viz.* $\frac{1}{2}BC + \frac{1}{2}bc = BC$.

These Things being premiſed, let us ſuppoſe the Triangle BCH to be a *Right-angled Triangle*, *viz.* the Side C perpendicular to the Side B; then will $BB + CC = HH$.

$B^2 + C^2 =$

Demonſtration.



Make a *Square* whoſe Side is $= B + C$, and draw the included *Square* whoſe Side is $= H$, as in the Sheme: Then will the *Area* of the *great Square* be equal to the *Area* of the Four *Triangles* $+ HH$, but the *Area* of each $\triangle = \frac{1}{2}BC$, per *Lemma* 3. Therefore the 4 $\triangle$'s $= \frac{1}{2}BC \times 4 = 2BC$. Conſequently, the *Area* of the great *Square* is $HH + 2BC$. *Involve* $B + C$, and it will be $BB + 2BC + CC$ = the *Area* of the great *Square*; per *Lemma* 3.

Conſequently, $HH + 2BC = BB + 2BC + CC$, per *Axiom* 5. Subſtract $2BC$ from both Sides of the *Æquation*, and there will *Remain* $HH = BB + CC$.

To illuſtrate this *Theorem* by *Numbers*, let us

Suppoſe $C = 3$. $B = 4$. and $H = 5$.
Then will $CC = 9$. $BB = 16$. and $HH = 25$.
Conſequently, $BB + CC = HH = 16 + 9 = 25$.

Conſectary.

From this admirable *Theorem* (*ſaid to be firſt invented by* Pythagoras) is deduced the *Method* of adding and ſubſtracting Squares, Parallelograms, Circles, *&c.*

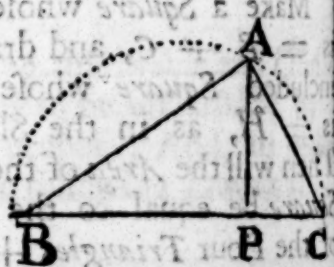
THEO-

THEOREM XII.

In any Right-angled Triangle, a Perpendicular being let fall from the Right-angle upon the Hypothenuse, will divide the Triangle into Two Right-angled Triangles, which will be both Similar (or alike) to the first Triangle, and to each other. (8. e. 6)

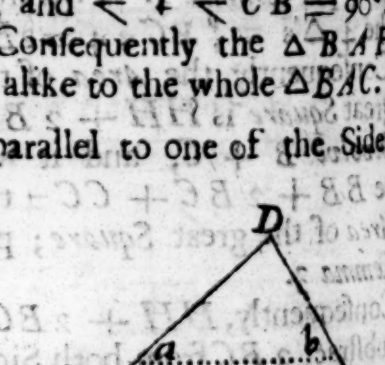
Note, *All plain Triangles are said to be Similar* (viz. *alike*) *when each single Angle in one of the Triangles is equal to each single Angle of the other; but if any two single Angles of one Triangle are equal to two single Angles of the other, the third Angle will be equal. Per* Theor. 4.

1. In the Right-angled $\triangle BAC$, let AP be supposed *Perpendicular* to the *Hypothenuse* BC; Then $\angle BAP = \angle C$.



For $\angle BAP + \angle B = 90°$,
And $\angle B + \angle C = 90°$, per *Corollary* to *Theorem* 4.
Therefore $\angle BAP = \angle C$. Per *Axiom* 5.
Again, $\angle PAC + \angle C = 90°$, and $\angle + \angle CB = 90°$.
Therefore $\angle PAC = \angle B$, &c. Consequently the $\triangle BAP$ is alike to the $\triangle APC$; and each is alike to the whole $\triangle BAC$.

2. Or, if a *Right-line* be drawn parallel to one of the Sides of any *plain Triangle*, (*viz.* within it) it will cut off a Triangle similar or alike to the whole Triangle. Thus,



In the $\triangle ABD$ draw the *Right-line* ab parallel to the Side AB; Then will the included $\triangle aDb$ be like the $\triangle ADB$: For $\angle a = \angle A$, and $\angle b = \angle B$. Per *Theorem* 3. and $\angle D$ is common to both the Triangles; *Ergo*, &c.

THEOREM XIII.

If two Triangles are alike, their like Sides will be proportional.

That is, those Sides which subtend the equal Angles, as also those Sides which are *about* the equal Angles, will be *proportional* to each other; and consequently, if any *two Triangles* have their Sides proportional, their Angles are equal. (4, 5, 6, 7. e. 6.)

Demon-

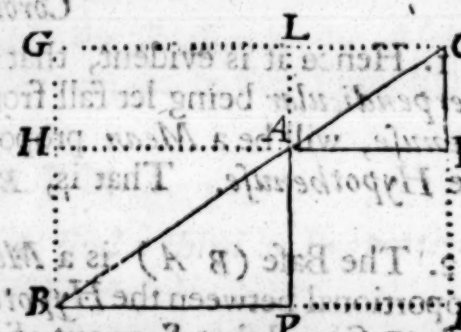
Demonstration.

Let the *Similar Triangles* in the *Scheme* of the last *Theorem* be here propos'd again.

Then it will be $BP : AP :: AP : CP$, according to this *Theorem*. *Ergo* $BP \times CP = AP \times AP$.

Demonstration.

Let us suppose the aforesaid *Right-angled* $\triangle BAC$ cut through the Perpendicular AP, and there open'd until the Sides BA and CA become one *Right-line*. Let the Sides BP and CP be continued until they meet in E; then compleat the Parallelograms by drawing the parallel Lines GLC, HAP, GHB, and LAP, as in the *Figure*.

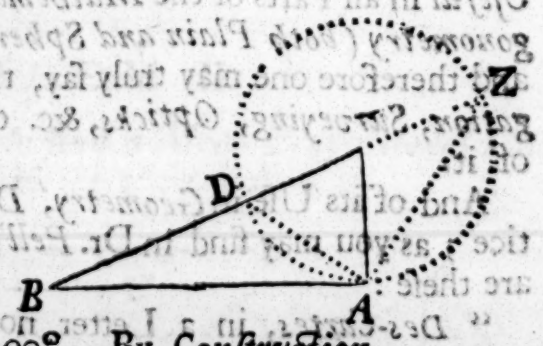


Then it is evident, that the $\triangle BHA = \triangle BPA$, and the $\triangle CPA = \triangle CLA$; also that the $\triangle BEC = \triangle BGC$, because all their respective Sides are *Equal*.

But the $\triangle BHA + \triangle CLA + \square HGLA = \triangle BPA$, $+ \triangle CPA + \square APEP$. Now, if from both *Sides* of this *Æquation* there be substracted the equal *Triangles*, there will remain $HGLA = BP \times CP$, and $\square AFEP = AP \times AP$. Consequently $BP : AP :: AP : CP$. Which was to be prov'd.

Or otherwise, thus:

Suppose the $\triangle BAC$ to be *Right-angled* at A, upon the $<$ Point C, with the *Radius* CA, describe a Circle, and continue the *Hypothenuse* BC to Z; joyn ZA and AD with *Right-lines*; then will the $\triangle BAD$ be like to the $\triangle BZA$.



For $< DAB + < BAC = 90°$. By *Construction*.

And $< ZAC + < DAC = 90°$. By *Theorem* X.

Therefore $< DAB + < DAC = < ZAC + < DAC$. By *Axiom* 5. *Substract* $< DAC$ from both sides of the *Æquation*, and there will remain $< DAB = < ZAC$. But $< ZAC = < CZA$,

By

By *Theorem* 6. And $\prec B$ is common to both Triangles. Therefore $\prec BDA = \prec BAZ$. By *Theorem* 6. Consequently $\triangle BAD$ is like to $\triangle BZA$.

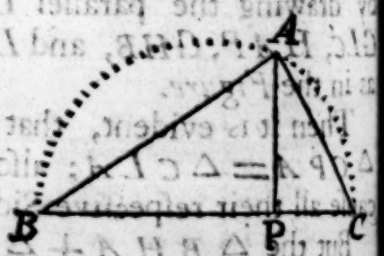
Let the *Sides* $\left\{\begin{matrix} BA = b \\ BC = b \\ CA = c \end{matrix}\right\}$ Then $bb + cc = hh$, by *Theorem* 11. Consequently $bb = hh - cc$, which gives the following Analogy,

Viz. $b : h + c :: h - c : b$; That is, $BA : BZ :: BD : BA$. Q. E. D.

Corollaries.

1. Hence it is evident, that in any *Right-angled Triangle*, a *Perpendicular* being let fall from the *Right-angle* upon the *Hypothenuse*, will be a *Mean* proportional between the *Segments* of the *Hypothenuse*. That is, $BP : PA :: PA : PC$.

2. The Base (BA) is a *Mean* Proportional between the *Hypothenuse* (BC) and that *Segment* of the *Hypothenuse* next to the Base, (viz. BP) that is, $BC : BA :: BA : BP$.



3. The *Cathetus* (AC) is a *Mean* Proportional between the *Hypothenuse* (BC) and that *Segment* of the *Hypothenuse* next to the *Cathetus*, (viz. PC): That is, $BC : AC :: AC : PC$.

Scholium.

I have been more Large upon this most excellent *Theorem*, in giving a double *Demonstration* of it, because 'tis so *Universally Useful* in all Parts of the *Mathematicks*: For the Business of *Trigonometry (both Plain and Spherical)* wholly depends upon it; and therefore one may truly say, that *Astronomy, Dyalling, Navigation, Surveying, Opticks*, &c. depend upon a due Application of it.

And of its Use in *Geometry*, *Des-Cartes* takes particular Notice; as you may find in Dr. *Pell*'s Algebra, *p.* 65, whose Words are these:

" *Des-Cartes*, in a Letter not yet printed, writes thus: In " searching the *Solution* of *Geometrical Questions*, I always make " use of Lines Parallel and Perpendicular, as much as is possible, " [*be means as many Lines as are useful*] and I consider no " other Theorems but these Two, [*the Sides of like* Trian- " gles *have like Proportion*]. And [*in Rectangle Triangles*

" *the*

" *the Square of the greateſt Side is equal to the Squares of the* " *two other Sides.*] And I am not afraid to ſuppoſe many Un- " known Quantities, that I may reduce the propos'd Queſtion to " ſuch Terms, as to depend on no other Theorems but theſe " Two.

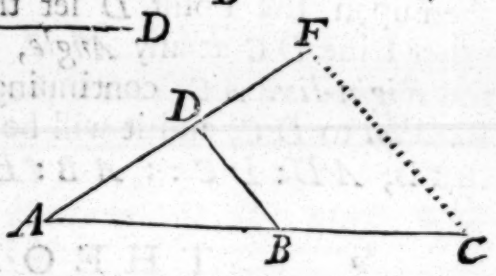
This I thought convenient to inſert, that the young *Learner* may ſee how the Great *Des-Cartes* eſteem'd theſe two Theorems, *viz.* the laſt, and Theorem 11; for, in truth, all the precedent Theorems are only (as it were) Preparatives to theſe Two.

This laſt Theorem demonſtrates the *Reaſon* of the *Method* uſed in finding out *Proportional Lines*; as in the Three following *Problems*.

PROBLEM I.

Two Right-lines being given, to find a third in Proportion to them. (11. *e.* 6.)

Let the two *Lines* be $\begin{cases} A \text{——} B \\ A \text{——} D \end{cases}$

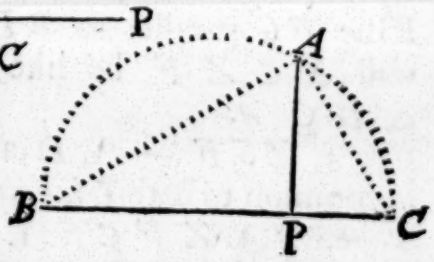


Set the Two given *Lines* at any Angle in the Point *A*, and produce the Line *A B* to *C*, making *B C* = *A D*; joyn the Points *B D* with a *Right Line*, and draw *C F* Parallel to *B D*; then will the △ *A B D* be like the △ *A C F*. Therefore *A B* : *B C* (= *A D*) :: *A D* : *D F*, which is the third *Proportional* requir'd.

PROBLEM II.

Two Right-lines being given, to find a Mean Proportional Line between them. (13. *e.* 6.)

Let the given *Lines* be $\begin{cases} B \text{——} P \\ P \text{——} C \end{cases}$



Joyn the Two given *Lines* into one, *viz.* make *B C* = *B P* + *P C*, and upon *B C*, as *Diameter*, deſcribe a *Semicircle*; then upon the Point *P*, where the Two *Lines* meet, erect a Perpendicular to touch the

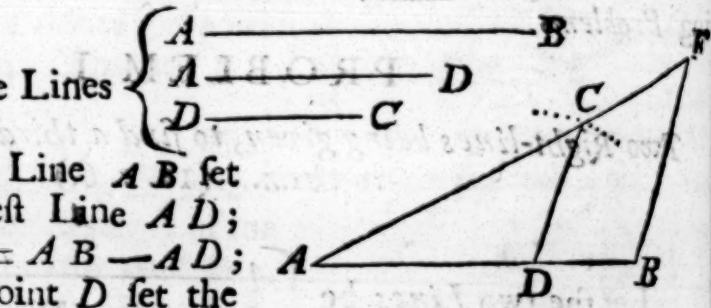
Circle's *Periphery*, as *P A*, and it will be the *Mean Proportional* requir'd, viz. *B P* : *A P* :: *A P* : *P C*.

By this *Problem* 'tis easie to conceive how to make a *Square* equal to any given *Parallelogram*. (14. *e.* 6.)

For if *B P* be the *length*, and *P C* the *breadth* of the given *Parallelogram*, then will *A P* be the Side of the *Square*, equal in *Area* to that *Parallelogram*.

PROBLEM III.

The Right-lines being given, to find a Fourth Proportional Line. (12. *e.* 6.)

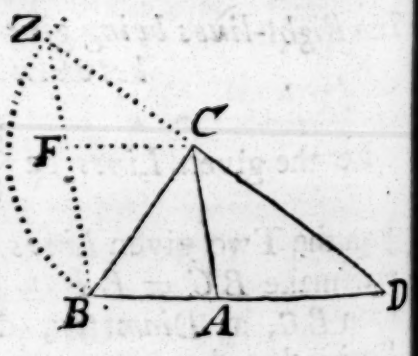


Suppose the Three Lines *A B*, *A D*, *D C*. Upon the longest Line *A B* set off the next longest Line *A D*; *viz.* make *D B* = *A B* — *A D*; then upon the Point *D* set the other Line *D C* at any *Angle*, either *Right* or *Oblique*, and draw the *Right-line A C*, continuing it a sufficient length; make *B F Parallel* to *D C*, and it will be the Fourth *Proportional* requir'd; that is, *A D* : *D C* :: *A B* : *B F*.

THEOREM XIV.

If any Angle of a plain Triangle be bisected (viz. *divided into two equal Angles*) *with a Right-line*, (viz. *as C A is supposed to do the Angle B C D*) *it will cut the opposite Side* (viz. *B D*) *in proportion to the other two Sides of the Triangle.* (3. *e.* 6.)

Demonstration.



Produce the Side *D G*, until *C Z* = *C B*; joyn the Points *Z B* with a *Right-line*, and draw the Line *F C* parallel to *B D*; then will △ *C Z F* be like to △ *D C A*.

For ∠ *Z C F* — ∠ *D* and ∠ *Z* is common to both *Triangles*, consequently ∠ *Z F C* = ∠ *C A D* and *F C* — *B A*.

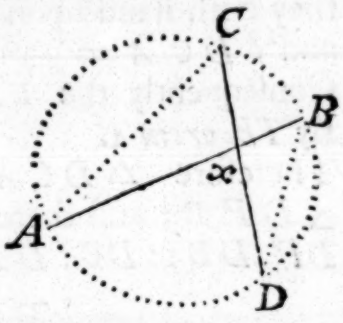
Therefore *B A* (= *F C*) : *B C* (= *Z C*) :: *A D* : *C D*. Q. E. D.

THEO-

THEOREM XV.

If two Right-lines (howsoever drawn) within a Circle *do cut each other, the Rectangle made of the Segments (or Parts) of the one Line, will be Equal to the Rectangle made of the Segments (or Parts) of the other Line.* (35. *e.* 3)

That is, if two Lines (as *A B* and *C D*) do cut each other in any Point, as at *x*, then will $A x \times B x = D x \times C x$.



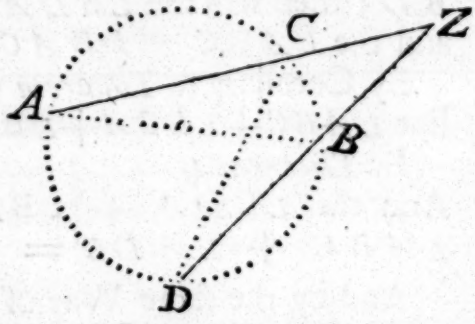
Demonstration.

Joyn the Points *A C* and *B D* with *Right-lines*, then will the $\triangle C x A$ be like to the $\triangle B x D$: For $\angle B = \angle C$ and $\angle A = \angle D$. By Corollary to Theorem 9.
And $\angle A \times C = \angle B \times D$. By *Theorem* 2.
Therefore it will be $A x : D x :: C x : B x$. By *Theorem* 13.
Consequently $A x \times B x = D x \times C x$. Q. E. D.

THEOREM XVI.

If two Right-lines are so drawn within a Circle *as, being continued, they will meet in a Point out of the* Circle's *Periphery, the Rectangle made of one whole Line, and its Part out of the* Circle, *will be Equal to the Rectangle of the other whole Line, and its Part out of the* Circle. (36, 37. *e.* 3.)

That is, if the Lines *A C* and *D B* be continued unto the Point *Z*;
Then will $A Z \times C Z = D Z \times B Z$.



Demonstration.

Draw the Lines *A B* and *C D*, then will $\triangle C Z D$ be like to the $\triangle B Z A$; for $\angle A = \angle D$ and $\angle Z$ is common to both *Triangles*. Consequently, $\angle A B Z = \angle D C Z$. By *Theorem* 4.
Therefore $A Z : B Z :: D Z : C Z$. *Ergo*, $A Z \times C Z = D Z \times B Z$.

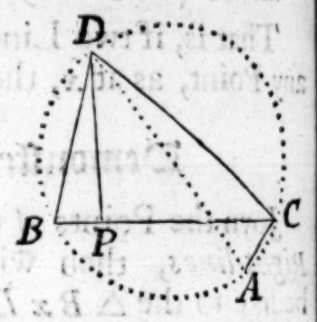
THEOREM XVII.

If from any Angle of a plain Triangle inscrib'd in a Circle *there be let fall a Perpendicular upon the opposite Side, (as D P)*

as that Perpendicular is in proportion to one of the Sides including the Angle, ſo is the other Side including the Angle to the Diameter of the Circle.

Demonſtration.

Let *B C D* be the propos'd *Triangle*. From the ∠ at *D* draw the Diameter *D A*; Then will ∠ *A* = ∠ *B*, becauſe they both ſtand upon the ſame Arch *D C* and ∠ *D C A* = 90°. By *Theorem* 10. Conſequently the ∠ *A D C* = ∠ *B D P*. By *Theorem* 4.



Therefore △ *D C A* is like to the △ *D P B*; and therefore,
D P : *D B* :: *D C* : *D A*; or, *D P* : *D C* :: *D B* : *D A*. Q. E. D.

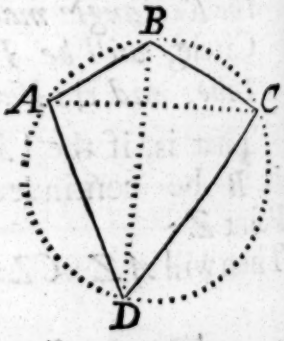
THEOREM XVIII.

If any Quadrangle (that is, a Trapezium*) be inſcrib'd within a* Circle, *the two oppoſite Angles, taken together, are Equal to two Right-angles,* viz. 180°. (22. *e.* 3.)

That is, in the *Quadrangle A B C D* the ∠ *A* + ∠ *C* = 180°. And the ∠ *B* + ∠ *D* = 180°.

Demonſtration.

Draw the *Two* Diagonals *A C*, and *B D*; then will the ∠ *B D A* = ∠ *B C A*, and the ∠ *A B C* = ∠ *B A C*.



By Corollary to *Theorem* 9.
But ∠ *A B C* + ∠ *B C A* + ∠ *B A C* = 180°.
By *Theorem* 4.
And the ∠ *B D A* + ∠ *B D C* = ∠ *A D C*. Therefore the ∠ *A B C* + ∠ *A D C* = 180°.

And by the ſame Way of Arguing it may be prov'd, that the ∠ *B A D* + ∠ *B C D* = 180°. Q. E. D.

THEOREM XIX.

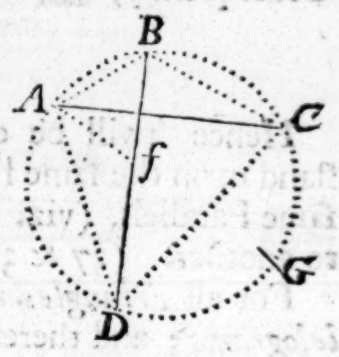
If in any Quadrangle inſcrib'd within a Circle *there be drawn two Diagonals, as A C and B D, the Rectangle made of the two Diagonals will be Equal to both the Rectangles made of the oppoſite Sides of the Quadrangle.*

That is, *A C* × *B D* = *A B* × *C D* + *A D* × *B C*.

Demon-

Demonstration.

Make the Arch DG = Arch BC, and from the Points AG draw the Line Af, and it will form the $\triangle Af\,D$, like to the $\triangle ABC$: For the $\angle fAD = \angle BAC$, because the Arches DG and BC are Equal.



Again, the $\angle fDA = \angle BCA$, because they both stand upon the Arch AB: Consequently, the $\angle AfD = \angle ABC$. By *Theorem* 4.

Therefore it will be $AC : BC :: AD : Df$. By *Theorem* 13.

$$Ergo \frac{BC \times AD}{AC} = Df$$

Again, the $\triangle BAf$ and $\triangle ACD$ are alike: For $\angle ABf = \angle ACD$, and $\angle BAf = \angle CAD$, because the $\angle fAD = \angle BAC$. And the $\angle CAf$ is common to both *Triangles*. Consequently, the $\angle AfB = \angle ADC$.

Therefore $AC : CD :: AB : Bf$. By *Theorem* 13.

$$Ergo \frac{CD \times AB}{AC} = Bf. \text{ But } Df + Bf = BD.$$

Consequently, $BC \times AD + CD \times AB = BD \times AC$.

Q. E. D.

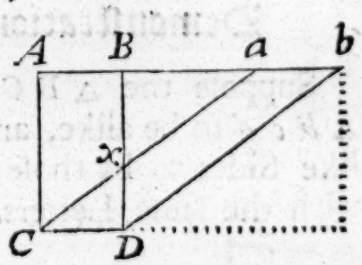
THEOREM XX.

All Parallelograms (whether Right or Oblique-angled) that stand upon the same Base, *or upon equal* Bases, *and betwixt the same Parallels, are Equal to one another.* (35 & 36. *e.* 1.)

That is, $\square ABCD = \square abCD$.

Demonstration.

Because $AB = CD = ab$ by Supposition, therefore $Aa = Bb$; for Ba is common to both. And because $AC = Bd$, and the $\angle A = \angle B$, therefore the $\triangle ACa = \triangle BDb$: And if from both Triangles there be taken the $\triangle Bxa$ common to both, there will remain the *Trapeziums* $ABxC = abxD$. Per *Axiom* 5.



But

But *Trapezium* $A B \times C + \triangle C \times D = \square A B C D$.
And *Trapezium* $a b \times D + \triangle C \times D = \square a b C D$.
Conſequently, $\square A B C D = \square a b C D$. Q. E. D

Corollary.

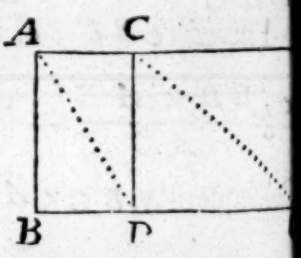
Hence 'twill be eaſie to conceive, that all *Triangles* which ſtand upon the ſame Baſe, or upon Equal Baſes, and between the ſame Parallels, (viz. *having the ſame Heighth*) are Equal one to another. (37 & 38. *e.* 1.)

For all *Triangles* are the Halfs of their Circumſcribing *Parallelograms*; and therefore, if the Wholes be Equal, their Halfs will alſo be Equal.

THEOREM XXI.

Parallelograms (*and conſequently* Triangles) *which have the ſame Heighth, have the ſame Proportion one to another as their* Baſes *have.* (1. *e.* 6.)

Demonſtration.



Draw $A F$ Parallel to $B G$, and draw $A B$, $C D$, $F G$ Perpendiculars to them.
Then will $B D \times A B \} = \square A B C D$.
And becauſe $C D = A B$, therefore $D G \times A B = \square C D F G$. But $B D : D G :: B D \times A B : D G \times A B$.
And conſequently $\triangle A B D : \triangle C D G :: B D : D G$, &c.
Q. E. D

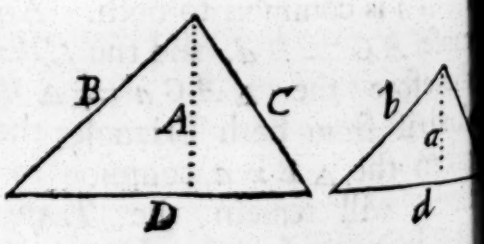
THEOREM XXII.

Like Triangles are in duplicate Ratio *to that of their homologous Sides.* (19. *e.* 6.)

That is, The *Area's* of like *Triangles* are in *Proportion* one to another as are the *Squares* of their like Sides.

Demonſtration.

Suppoſe the $\triangle B C D$ and $\triangle b c d$ to be alike, and their like Sides to be thoſe mark'd with the ſame Letters.



Let A and a be Perpendiculars to the two Bases D and d.

Then $\frac{1}{2} D A$ = the Area of $\triangle B C D$ } By *Lemma* 3, *page* 303.
And $\frac{1}{2} d a$ = the Area of $\triangle b c d$ }

But	1	$B : b :: D : d$ } &c. By *Theorem* 13.
And	2	$B : b :: A : a$ }
Conseq.	3	$D : d :: A : a$
3 ·|·	4	$D a = d A$
$\times \frac{1}{2} Dd$	5	$\frac{1}{2} D D d a = \frac{1}{2} D d d A$. By *Axiom* 3.
Hence	6	$D D : d d :: \frac{1}{2} D A : \frac{1}{2} d a$. And so for other Sides.

Q. E. D.

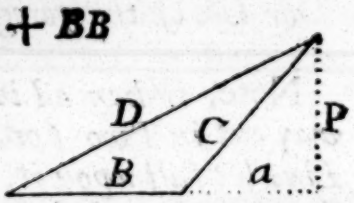
THEOREM XXIII.

In every Obtuse-angled Triangle (as B C D) the Square of the Side subtending the Obtuse Angle (as D) is greater than the Squares of the other two Sides (B and C) by a double Rectangle made out of one of the Sides (as B) and the Segment or Part of that Side produc'd, (as a) until it meet with the Perpendicular (P) let fall upon it. (12. e. 2.)

That is, $DD = BB + CC + 2Ba$.

Demonstration.

First	1	$DD = PP + aa + 2Ba + BB$
And	2	$CC = PP + aa$
1 − 2	3	$DD - CC = 2Ba + BB$
1 + CC	4	$DD = BB + CC + 2Ba$



Q. E. D.

Corollary.

Hence 'tis evident, that if the Sides of any *Obtuse-angled Triangle* are given, the *Segment* (*a*) of the Side produc'd (or the Perpendicular, P) may be easily found.

THEOREM XXIV.

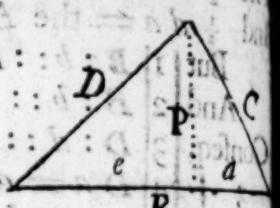
If a Perpendicular (*as* P) *be let fall in any Acute-angled Triangle* (*as B C D*) *the Square of either of the Two Sides* (*as D*) *is less than the Squares of the other Side, and that Side upon which the* Perpendicular *falls* (viz. C *and* B) *by a double Rectangle made of the Side* B, *and that Segment or Part of it* (viz. *a*) *which lies next to the Side C.* (13. e. 2.)

That is, $DD + 2Ba = BB + CC$.

Demon-

Demonstration.

First	1	$DD = PP + ea$	By *Theor.* 11.
And	2	$CC = PP + aa$	
But	3	$B - a = e$, by Figure.	
3 × 2	4	$BB - 2Ba + aa = ee$.	
4 − aa	5	$BB - 2Ba = ee - aa$.	
1 − 2	6	$DD - CC = ee - aa$.	
5, 6	7	$DD - CC = BB - 2Ba$.	
7 ±	8	$DD + 2Ba = BB + CC$	Q. E. D.



Corollary.

Hence it follows, that if the Sides of any *Acute-angled Triangle* be known, the Perpendicular *P*, and the Segments of the Side whereon it falls (*viz.* *a* *e*) may be easily found.

CHAP. IV.

The Solution *of several Easie* Problems *in Plain Geometry whereby the Learner may (in part) perceive the Application or Use of the foregoing Theorems.*

Note, *When a Line, or the Side of any plain Triangle, is any way cut in Two (or more) Parts, either by a Perpendicular Line let fall upon it, or otherwise, those Parts are usually call'd* Segments; *and so much as one of those Parts is Longer than the other, is call'd the* Difference of the Segments.

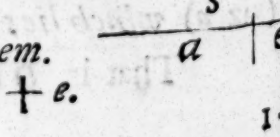
And when any Side of a Triangle, or any Segment of its Side is given, 'tis usually mark'd with a small Line cross it, thus —|— *and those Sides, or Parts of Sides, that are sought, are marked with four Points, thus* ——.

PROBLEM I.

To cut or divide a given Right-line (as S) into Extream and Mean Proportion. (11. *e.* 2.)

THat is, to *divide* a *Line* so, that the *Square* of the *greater Segment* (or *Part*) *a*, may be Equal to the *Rectangle* made of the whole Line *S*, and the lesser Segment *e*.

Viz.	1	$Se = aa$, by the *Problem*.	
And	2	$S - a = e$, for $S = a + e$.	



1 ÷ S	3	$\frac{aa}{S} = e$
2 and 3	4	$\frac{aa}{S} = S - a$. By *Axiom* 5.
4 × S	5	$aa = SS - Sa$
5 + Sa	6	$aa + Sa = SS$
6, ſolved	7	$a = \sqrt{SS + \frac{1}{4}SS} : \frac{1}{2}S$. See *Pages* 195, 196.

Note, *The laſt* Problem *cannot be truly anſwer'd by Numbers, but Geometrically it may be perform'd, thus:*

1. Make a Square, whoſe Side is $= S$ the given Line, and biſect one of its Sides in the middle, as at C; upon the Point C deſcribe ſuch a Semicircle as will paſs through the remoteſt Points of the Square, and compleat its Diameter.

2. Then will either Part of the Diameter, on each End of the Side S, be $= a$, the *greater Segment* ſought.

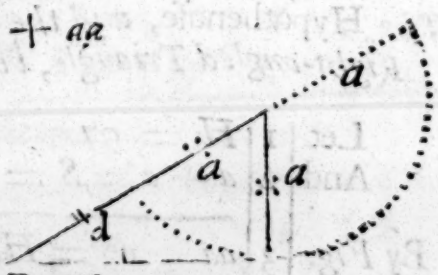
But $a + S : S :: S : a$. By *Theorem* 13.

Ergo, $aa + Sa = SS$. Which was to be done.

PROBLEM II.

The Baſe *of any Right-angled Triangle, and the difference between the* Hypothenuſe *and* Cathetus *being given, To find the* Cathetus, *&c.*

Let	1	$b = 72$
	2	$d = 32$
And	3	$a =$ *Cathetus* ſought
Then	4	$\overline{bb + aa} = \overline{dd + 2da + aa}$ By *Theorem* 11.
4 − aa	5	$bb = dd + 2da$
5 − dd	6	$2da = bb - dd$
6 ÷ $2d$	7	$a = \frac{bb - dd}{2d} = 65$
Or,	8	$b : d + 2a :: d : b$. By *Theorem* 13.
8 ∴	9	$bb = dd + 2da$. As before at the 5th Step.

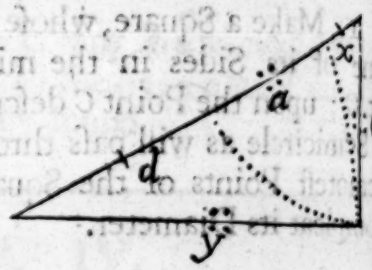


Here you see that either Way raises the same *Æquation*; neither is there any constant *Method* or *Road* to be observ'd in solving *Geometrical Problems*, but every one makes use of such Ways and Theorems as happen to come first into their Mind, the Result being every Way the same.

PROBLEM III.

The difference between the Base *and* Hypothenuse *of any Right-angled Triangle, and the difference between the* Cathetus *and* Hypothenuse *being both given, To find the Triangle.*

Let	1	$d = 32$
	2	$x = 25$
And	3	$d + x + a =$ the *Hypot.*
Then	4	$d + a = y$
	5	$x + a = e$ } by *Probl.*
4 ⊙ 2	6	$dd + 2da + aa = yy$
5 ⊙ 2	7	$xx + 2xa + aa = ee$
3 ⊙ 2	8	$dd + 2dx + 2da + 2xa + xx + aa = \square$ *Hypothenuse.*
6 + 7	9	$dd + 2da + 2xa + xx + 2aa = yy + ee$



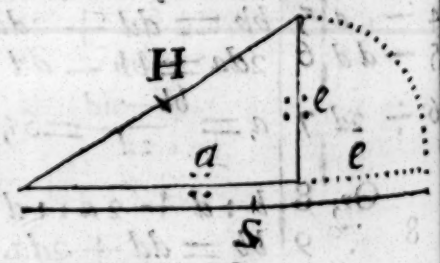
The two last Steps are Equal, by *Theorem* 11. Consequently if those Things that are Equal in both be taken away, the Remainders will be Equal. By *Axiom* 2.

That is,	10	$aa = 2dx = 1600$	
10 ω 2	11	$a = \sqrt{2dx} = 40$	
1 + 11	12	$d + a = 72 = y$	*The* Base.
2 + 11	13	$x + a = 65 = e$	*The* Cathetus.
1 + 2 + 11	14	$d + x + a = 97$	*The* Hypothenuse.

PROBLEM IV.

The Hypothenuse, *and the Sum of the other two Sides of any Right-angled Triangle, being given, Thence to find the Sides.*

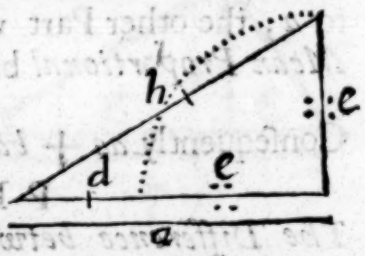
Let	1	$H = 97$
And	2	$a + e = S = 137$
By *Fig.*	3	$aa + ee = HH$
2 ⊙ 2	4	$aa + 2ae + ee = SS$
4 — 3	5	$2ae = SS - HH$
3 — 5	6	$aa - 2ae + ee = 2HH - SS$
6 ω 2	7	$a - e = \sqrt{2HH - SS}$



2 + 7	8	$2a = S + \sqrt{2HH - SS} = 144$
8 ÷ 2	9	$a = \frac{S + \sqrt{2HH - SS}}{2} = 72$ *The* Base *requir'd.*
2 − 9	10	$e = \frac{S - \sqrt{2HH - SS}}{2} = 65$ *The* Cathetus

PROBLEM V.

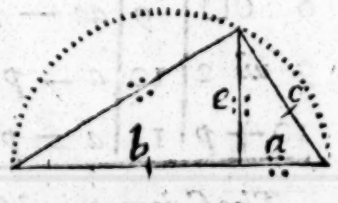
The Hypothenuse, *and the Difference of the other two Sides of any Right-angled Triangle being given, To find the Sides.*



Let	1	$h = 97$ As before.
And	2	$a - e = d = 7$ Quere a
By *Fig.*	3	$aa + ee = hh$
2 ⊙ 2	4	$aa - 2ae + ee = dd$
3 − 4	5	$2ae = hh - dd$
3 + 5	6	$aa + 2ae + ee = 2hh + dd$
6 ww 2	7	$a + e = \sqrt{2hh - dd}$
2 + 7	8	$2a = d + \sqrt{2hh - dd} = 144$
8 ÷ 2	9	$a = 72$
7 − 2	10	$2a = \sqrt{2hh - dd} : - d = 130$
1 ÷ 2	11	$e = 65$

PROBLEM VI.

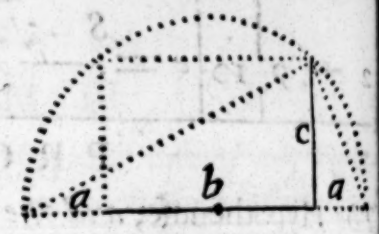
In any Right-angled Triangle, either the Base, *or* Cathetus, *and the Alternate Segment of the* Hypothenuse, *(made by a Perpendicular let fall from the Right-angle) being given, To find the other Segment.*



Let	1	$c = 45$ *The* Cathetus
And	2	$b = 48$ *The* alternate *Segm.*
Then	3	$b : e :: e : a$ Quere a
3 ∴	4	$ba = ee$
Again,	5	$cc - aa = ee$. By *Theor.* 11.
4, 5	6	$ba = cc - aa$
6 + aa	7	$aa + ba = cc$
7, C□	8	$aa + ba + \frac{1}{4}bb = cc + \frac{1}{4}bb$
8 ww 2	9	$a + \frac{1}{2}b = \sqrt{cc + \frac{1}{4}bb}$
9 − $\frac{1}{2}b$	10	$a = \sqrt{cc + \frac{1}{4}bb} : - \frac{1}{2}b = 27$ And so on for e, &c.

I shall now shew the *Geometrical Construction* (or *Solution*) of the Three Cases of *Quadratick Æquations* promis'd in Page 202. Let the first Example be that above, *viz.* $aa + ba = cc$. Case 1.

Make the Co-efficient b, and the *Root* of the *Resolvend* (which is here) c, into a *Right-angled Parallelogram*. And upon the middle Point of the Side $= b$ describe such a Semicircle as will pass thro' the remotest Points or Angles of the *Parallelogram*, compleating its Diameter as in the annex'd *Scheme*.

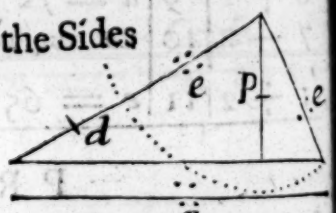


Then will either Part of the Diameter on each End b, be Equal to a; the other Part will be $a + b$, and the Side c will be a *Mean Proportional* between them: That is, $a + b : c :: c : a$. By *Theorem* 13.

Consequently $aa + ba = cc$. Which was to be done.

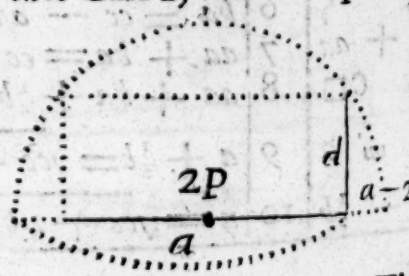
PROBLEM VII.

The Difference between the Base *and* Cathetus *of any Right-angled Triangle, and the* Perpendicular *let fall from the Right-angle upon the* Hypothenuse, *being given; Thence to find the* Hypothenuse, *&c.*



Let	1	$d = 15$ The Difference of the Sides
And	2	$p = 36$
Quere a	3	$a =$ *The* Hypothenuse.
By *Fig.* 4	4	$d + e : p :: a : e$
∴	5	$de + ee = pa$
Again,	6	$dd + 2de + 2ee = aa$ By *Theorem* 11.
5 × 2	7	$2de + 2ee = 2pa$
6 — 7	8	$dd = aa - 2pa$ Case 2.
8 $C\square$	9	$aa - 2pa + pp = dd + pp = 1521.$
9 ww 2	10	$a - p = \sqrt{dd + pp} = 39$
10 + p	11	$a = p + \sqrt{dd + pp} = 75$, &c. for e. per *Step* 5.

The Geometrical Construction of this Case 2, viz. $aa - 2pa = dd$ may be perform'd in the very same manner as the *last Case* was; that is, by making a Right-angled Parallelogram of the Co-efficient $2p$ and the $\sqrt{dd}$, viz. d, &c. As in the annexed Figure.



Then

Then will the *Greater* Part of the Diameter to one End of the *Parallelogram* be $= a$, and the *Lesser* Part will be $a - 2p$

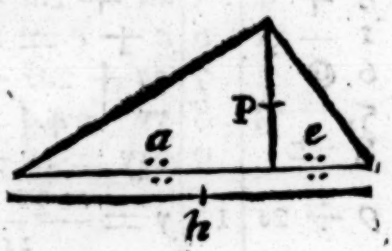
For $a : d :: d : a - 2p$ By *Theorem* 13.

Consequently, $aa - 2pa = dd$. Which was to be done.

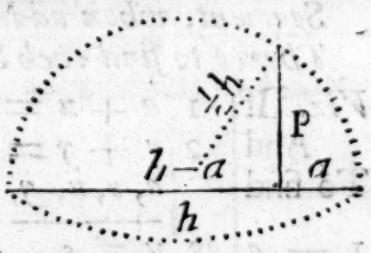
PROBLEM VIII.

The Hypothenuse *of any Right-angled Triangle, and the* Perpendicular *let fall from the Right-angle upon the* Hypothenuse, *being given, To find the greater Segment of the* Hypothenuse, &c.

Let	1	$h = 75$	*The* Hypothenuse
And	2	$p = 36$	
Then	3	$a + e = h$	*Quere a*
per *Fig.*	4	$a : p :: p : e$	
4 ::	5	$\frac{pp}{a} = e$	
3 − a	6	$h - a = e$	
5, 6	7	$h - a = \frac{pp}{a}$	
7 × a	8	$ha - aa = pp$	Case 3.
8 ±	9	$aa - ha = -pp$	
9 C□	10	$aa - ha + \frac{1}{4}hh = \frac{1}{4}hh - pp = 110, 25$	
10 ww 2	11	$a - \frac{1}{2}h = \sqrt{\frac{1}{4}hh - pp} = 10, 5$	
11 + $\frac{1}{2}h$	12	$a = \frac{1}{2}h \pm \sqrt{\frac{1}{4}hh - pp} = 48$. Or, $a = 27$.	



The Geometrical Construction of *Case* 3, viz. $ha - aa = pp$, may be thus perform'd: Draw a *Right Line* (of any convenient length at pleasure) and near its middle erect a Perpendicular $= p$, *viz.* of the same length with the *Root* of the *Resolvend*. From the top-Point or upper-End of that Perpendicular, set off half the length of the Co-efficient, *viz.* $\frac{1}{2}h$, and upon the Point where $\frac{1}{2}h$ just touches the first Line (with the same Distance) describe a Semicircle; then will its Diameter h be cut by the Perpendicular p into two Segments, which are the two Values of the Root a, viz, the *greater* and *lesser Roots*, both taken together, being always Equal to the Co-efficient; (*vide pag.* 201.)



For $h - a : p :: p : a$ By *Theorem* 13.

Ergo, $ha - aa = pp$. Which was to be done.

PRO-

PROBLEM IX.

The Perimeter, *viz. the Sum of all the three Sides of any Right-angled Triangle, and its* Area, *being given, Thence to find each Side.*

Viz. Let	1	$a + e + y = s = 234$	The *Sum* of the *Sides.*
And	2	$\frac{1}{2}ae = A$	The *Area* $= 2340$
Again,	3	$aa + ee = yy$	By Figure
2 × 4	4	$2ae = 4A$	
3 + 4	5	$aa + 2ae + ee = yy + 4A$	
1 − y	6	$a + e = s - y$	
6 ⊙ 2	7	$aa + 2ae + ee = ss - 2sy + yy$	
5, 7	8	$yy + 4A = ss - 2sy + yy$	
8 ±	9	$2sy = ss - 4A = 45396$	
9 ÷ $2s$	10	$y = \frac{ss - 4A}{2s} = \frac{1}{2}s - \frac{2A}{s} = 97$	*The* Hypothenuse.
6, 10	11	$a + e = s - y = 137$	
3 − 4	12	$aa - 2ae + ee = yy - 4A = 49$	
12 ww 2	13	$a - e = \sqrt{49} = 7$	
11 + 13	14	$2a = 137 + 7 = 144$	
13 ÷ 2	15	$a = 72$	*The* Base.
11 − 15	16	$e = 137 - 72 = 65$	*The* Cathetus.

PROBLEM X.

In any Right-angled Triangle a Perpendicular *being let fall from the Right-angle upon the* Hypothenuse, *if the Sum of each Segment, when added to its adjacent or next Side, be given. Thence to find each Side, and the Segments.*

Viz. If	1	$a + u = s$	108
And	2	$e + y = z$	$= 72$
To find		$a, e, u, y,$ and p	
1 − a	3	$u = s - a$	
3 ⊙ 2	4	$uu = ss - 2sa + aa$	
4 − aa	5	$uu - aa = ss - 2sa = pp$	
2 − e	6	$z - e = y$	
6 ⊙ 2	7	$zz - 2ze + ee = yy$	
7 − ee	8	$zz - 2ze = yy - ee = pp$	
5, 8	9	$zz - 2ze = ss - 2sa$	
By *Fig.*	10	$a : p :: p : e$	
10 ∴	11	$ae = pp$	
5, 11	12	$ae = ss - 2sa$	

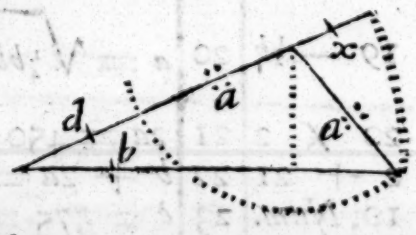
13 ÷

12 ÷ a	13	$e = \frac{ss - 2sa}{a}$
13 × $2z$	14	$2ze = \frac{2zss - 4zsa}{a}$
9 + 14	15	$zz = ss - 2sa + \frac{2zss - 4zsa}{a}$
15 × a	16	$zza = ssa - 2saa + 2zss - 4zsa$
16 ±	17	$2saa + zza + 4zsa - ssa = 2zss$
17 ÷ $2s$	18	$aa + \frac{zza}{2s} + 2za - \frac{1}{2}sa = zs$
Substitute	19	$2x = \frac{zz}{2s} = 2z - \frac{1}{2}s = 114$
Then	20	$aa + 2xa = zs = 7776$
20 C□	21	$aa + 2xa + xx = xs + xx = 11025$
21 ω 2	22	$a + x = \sqrt{zs + xx} = 105$
22 − x	23	$a = \sqrt{xs + xx} : - x = 48$
1 − 23	24	$u = 60 =$ *The* Base.
ger 13	25	$e = \frac{ss}{a} - 2s = 27$
2 − 25	26	$y = 45 =$ *the* Cathetus.
23 + 25	27	$a + e = 75 =$ *the* Hypothenuse.

PROBLEM XI.

The Difference of the Sides of any Oblique-angled plain Triangle, the Difference of the Segments of the Base, *and the Difference between the greater Side and the* Base, *being given, To find the* Base, *&c.*

Let	1	$d =$ the Difference of the Sides $= 405$
	2	$b =$ the Difference of the Segments $= 495$
	3	$x = 165$ the Differ. of the greater Side and Base.
And	4	$a =$ the least Side
Then	5	$d + a + x =$ the Base
And	6	$d + a + x : d + 2a :: d : b$ By *Theorem* 16.
6 ::	7	$db + ba + bx = dd - 2da$
7 ±	8	$2da - ba = db + bx - dd$
8 ÷ $2d - b$	9	$a = \frac{db + bx - dd}{2d - b} = \frac{118125}{215} = 375$
1 + 9	10	$d + a = 780 =$ *the* greatest Side.
3 + 10	11	$d + a + x = 945 =$ *the* Base.



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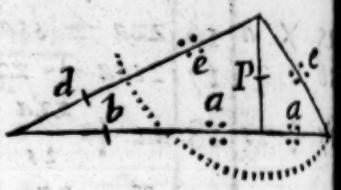
PROBLEM XII.

The Difference of the Sides of any plain Triangle; the Difference of the Segments of the Base, *and the* Perpendicular *let fall from the vertical Angle, being given, Thence to find all the Sides.*



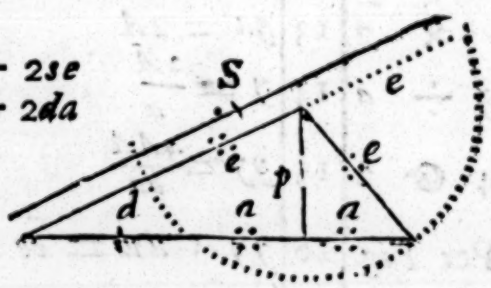
Let	1	$d = 405$
	2	$b = 495$ } as before.
And	3	$p = 300$
Quere	4	$a =$ the least *Segment*.
Then	5	$b + 2a : d + 2e :: d : b$
5	6	$bb + 2ba = dd + 2de$
6 — dd	7	$bb - dd + 2ba = 2de$
Substitute	8	$2x = bb - dd = 81000$
7, 8	9	$2x + 2ba = 2de$
9 ÷ $2d$	10	$\frac{x + ba}{d} = e$
But	11	$pp + aa = ee$ By *Theorem* 11.
10 @ 2	12	$\frac{xx + 2xba + bbaa}{dd} = ee$
11, 12	13	$\frac{xx + 2xba + bbaa}{dd} = pp + aa$
13 × dd	14	$xx + 2xba + bbaa = ppdd + ddaa$
14 ±	15	$bbaa - ddaa + 2xba = ppdd - xx$
8, 15	16	$2xai + 2xba = ppdd - xx$
16 ÷ $2x$	17	$aa + ba = \frac{ppdd}{2x} - \frac{1}{2}x$
17 C□	18	$aa + ba + \frac{1}{4}bb = \frac{1}{4}bb + \frac{ppdd}{2x} - \frac{1}{2}x$
18 ww 2	19	$a + \frac{1}{2}b + \sqrt{\frac{1}{4}bb + \frac{ppdd}{2x} - \frac{1}{2}x}$
19 — $\frac{1}{2}b$	20	$a = \sqrt{\frac{1}{4}bb + \frac{ppdd}{2} - \frac{1}{2}x} : -\frac{1}{2}b = 225$
20 × 2	21	$2a = 450$
2 + 21	22	$b + 2a = 945$ *the* Base.
10, *Num.*	23	$e = 375 =$ *the* lesser Side.
1 + 23	24	$d + e = 780 =$ *the* greater Side.

PROBLEM XIII.

The Sum of the two Sides of any plain Triangle, the Difference of the Segments of the Base, *and the* Perpendicular *let fall from the Vertex*

Vertical Angle upon the Base, *being given, Thence to find the* Base *and the Sides.*

Let	1	$s = 1155$ the *Sum* of the *Sides*.
	2	$d = 495$ the *Difference* of the *Segments*.
	3	$p = 300$ the *Perpendicular*.
Put	4	$a =$ the least *Segment*.
	5	$e =$ the least *Side*.
Then	6	$d + 2a =$ the *Base*.
And	7	$s - 2e =$ the *Difference* of the *Sides*.
Per *Fig.*	8	$d + 2a : s :: s : -2e : d$
	9	$aa + pp = ee$
9 ω 2	10	$\sqrt{aa + pp} = e$
8 ∴	11	$dd + 2da = ss - 2se$
11 ±	12	$2se = ss - dd - 2da$
Suppose	13	$2x = ss - dd$
Then	14	$2se = 2x - 2da$
14 ÷ 2s	15	$e = \frac{x - da}{s}$
10, 15	16	$\frac{x - da}{s} = \sqrt{aa + pp}$
16 ⊙ 2	17	$\frac{xx - 2xda + ddaa}{ss} = aa + pp$
17 × ss	18	$xx - 2xda + ddaa = ssaa + sspp$
18 ±	19	$ssaa - ddaa + 2xda = xx - sspp$
13, 19	20	$2xaa + 2xda = xx - sspp$
20 ÷ 2x	21	$aa + da = \frac{1}{2}x - \frac{sspp}{2x}$, &c. as before.
21, hence	22	$a = 225$
22 × 2	23	$2a = 450$
2 + 23	24	$d + 2a = 945$ the *Base*.
10, *Numb.*	25	$e = 375$ the lesser Side.
1 − 25	26	$s - e = 780$ the greater Side.

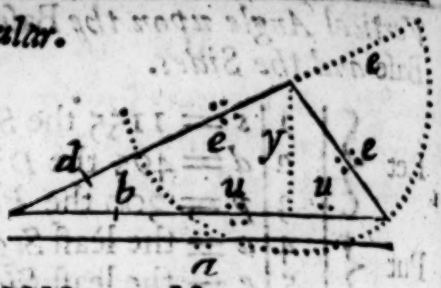


PROBLEM XIV.

The Area of any Oblique-angled plain Triangle, *the Difference of the Sides, and the Difference of the Segments of the* Base, *being given, Thence to find the* Base, &c.

Let	1	$A = 141750 =$ the *Area*.
	2	$d = 405$
	3	$b = 495$

Put {	4	y = the *Perpendicular*.
	5	a = the *Base*.
Then	6	$\frac{1}{2}ya = A$
Per *Fig.*	7	$a : d + 2e :: d : b$
7 ::	8	$ba = dd + 2de$
8 — dd	9	$ba - dd = 2de$
9 ⊙ 2	10	$bbaa - 2ddba + dddd = 4ddee$
Per *Fig.*	11	$\frac{a-b}{2} = u$ the leſſer Segment of the *Baſe*.
11 ⊙ 2	12	$\frac{aa - 2ba + bb}{4} = uu$
6 × 2	13	$ya = 2A$
13 ÷ a	14	$y = \frac{2A}{a}$
14 ⊙ 2	15	$yy = \frac{4AA}{aa}$
Per *Fig.*	16	$yy + uu = ee = \frac{4AA}{aa} + \frac{aa - 2ba + bb}{4}$
10 ÷ $4dd$	17	$\frac{bbaa - 2ddba + dddd}{4dd} = ee$
16, 17	18	$\frac{bbaa - 2ddba + d^4}{4dd} = \frac{4AA}{aa} + \frac{aa - 2ba + bb}{4}$
18 × aa	19	$\frac{bba^4 - 2ddba^3 + d^4aa}{4dd} = 4AA + \frac{a^4 - 2ba^3 + bbaa}{4}$
19 × $4dd$	20	$\begin{cases} bba^4 - 2ddba^3 + d^4aa = 16AAdd + dda^4 \\ \quad = 2ddba^3 + ddbbaa \end{cases}$
20 −	21	$bba^4 - dda^4 + d^4aa - ddbbaa = 16AAdd.$
21 ÷	22	$aaaa - ddaa = \frac{16AAdd}{bb - dd}$
22 C[]	23	$aaaa - ddaa + \frac{1}{4}dddd = \frac{16AAdd}{bb - dd} + \frac{1}{4}d^4$
23 ww 2	24	$aa - \frac{1}{2}dd = \sqrt{\frac{16AAdd}{bb - dd} + \frac{1}{4}d^4}$
24 + $\frac{1}{2}dd$	25	$aa = \frac{1}{2}dd + \sqrt{\frac{16AAdd}{bb - dd} + \frac{1}{4}dddd}$
25 ww 2	26	$a = \surd : \frac{1}{2}dd + \sqrt{\frac{16AAdd}{bb - dd} + \frac{1}{4}dddd} = 945$

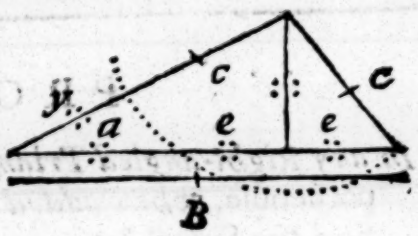


PRO-

PROBLEM XV.

There is an Oblique-angled plain Triangle, wherein a Perpendicular *is let fall from the Vertical Angle upon the* Base; *the least Side and the* Base *are given; and the Rectangle of the Difference of the Sides into the least Side is Equal to the Square of the Difference of the Segments of the* Base: *'Tis requir'd to find the Segments of the* Base, *&c.*

Let {	1	$c = 56 =$ the least Side.
	2	$B = 92 =$ the *Base*.
And	3	$a + 2e = B$
Put	4	$y =$ the Difference of the *Sides*.
Then	5	$cy = aa$ by the Question.
By *Figure* 6 ∴	6	$B : 2c + y :: y : a$, for $B = a + 2c$
	7	$Ba = 2cy + yy$
5 × 2	8	$2cy = 2aa$
7 − 8	9	$Ba - 2aa = yy$
5 ⊙ 2	10	$ccyy = aaaa$
10 ÷ cc	11	$yy = \frac{aaaa}{cc}$
9 11	12	$Ba - 2aa = \frac{aaaa}{cc}$
12 × cc	13	$ccBa - 2ccaa = aaaa$
13 ÷ a	14	$ccB - 2cca = aaa$
14 + $2cca$	15	$aaa + 2cca = ccB$
15, in *Num*	16	$aaa + 6272a = 288512$



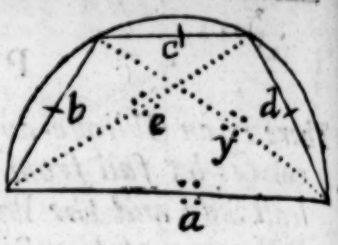
The *Value* of *a*, in this *Æquation*, may be found as in the Examples *page* 238, *viz.* by putting $r + e = a$, &c. as in those Examples you will find $a = 37{,}55502$, &c.

PROBLEM XVI.

The three Chords or Subtenses of three Arches compleating a Semicircle *being each given, Thence to find the Diameter of that* Circle. That is,

Any Trapezium *being inscrib'd in a* Semicircle, *if one of its Sides be the Diameter, and the other three Sides be given, Thence to find the Diameter, or Fourth Side.*

Let	1	$b = 3$
	2	$c = 4$ } the 3 Sides.
	3	$d = 5$
Quere	4	$a =$ the *Diam.* sought.



Draw the *two Diagonals* e and y

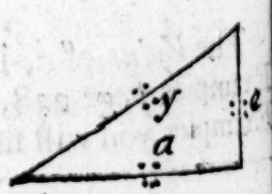
Then	5	$ca + bd = ey$ By *Theorem* 19.
And	6	$aa - bb = yy$ } By *Theorem* 10 & 11.
	7	$aa - dd = ee$
5 ⊙ 2	8	$ccaa + 2bdca + bbdd = eeyy$
6 × 7	9	$aaaa - bbaa - ddaa + bbdd = eeyy$
8, = 9	10	$aaaa - bbaa - ddaa = ccaa + 2bdca$
10 ÷ a	11	$aaa - bba - dda = cca + 2bdc$
11 − cca	12	$aaa - bba - dda - cca = 2bdc$
12, *Num.*	13	$aaa - 50a = 120$

This *Æquation* being solv'd, as in *Example* 2. *Page* 240, you will find $a = 8{,}05581$, &c.

PROBLEM XVII.

In any Right-angled Triangle, the Area *and the Sum of the* Hypothenuse, *when added to either Side, being given, Thence to find the Sides,* &c.

Suppose	1	$\frac{ae}{2} = A = 1350$ the *Area*
	2	$y + e = s = 120$ the *Sum,* &c.
	3	Quere a, e, and y
1 × 2	4	$ac = 2A$
4 ÷ a	5	$c = \frac{2A}{a}$
Per *Fig.*	6	$aa + ee = yy$
2 − e	7	$y = s - e$
5, 7	8	$y = s - \frac{2A}{a}$
8 ⊙ 2	9	$yy = ss - \frac{4sA}{a} + \frac{4AA}{aa}$
5 ⊙ 2	10	$ee = \frac{4AA}{aa}$
10 + aa	11	$aa + ee = \frac{4AA}{aa} + aa$



6, 9 11

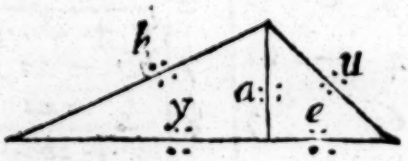
6, 9, 11	12	$\frac{4AA}{aa} + aa = yy = ss - \frac{4sA}{a} + \frac{4AA}{aa}$
12, That is	13	$aa = ss - \frac{4sA}{a}$
13 × a	14	$aaa = ssa - 4sA$
14 +	15	$ssa - aaa = 4sA$
15, in *Num*	16	$144000a - aaa = 648000$

The *Value* of a, in this *Æquation*, may be found as in the third Example, *page* 241; that is, by making $r + e = a$, &c. it will be found that $a = 60$.

PROBLEM XVIII.

There is an Oblique-angled plain Triangle, *wherein a* Perpendicular *is let fall from the* Vertical Angle *upon the* Base; *the Sum of each Segment of the* Base, *when added to its adjacent or next Side, and the* Area *of the* Triangle, *are given, To find the* Perpendicular, *and each Side.*

Let	1	$y + b = z = 1500$ } Quere y, b, e, and u.
	2	$e + u = s = 600$ }
	3	$A =$ the *Area* $= 141750$
And	4	$a =$ the *Perpendicular* sought.
Then	5	$y + e \times \frac{1}{2}a = A$
5 × 2 ÷ a	6	$y + e = \frac{2A}{a}$
Per *Fig.*	7	$yy + aa = bb$
	8	$ee + aa = uu$
1 − y	9	$b = z - y$
2 − e	10	$u = s - e$
9 ⊙ 2	11	$bb = zz - 2zy + yy$
10 ⊙ 2	12	$uu = ss - 2se - ee$
7, 11	13	$zz - 2zy = aa$
8, 12	14	$ss - 2se = aa$
13 ±	15	$zz - aa = 2zy$
14 ±	16	$ss - aa = 2se$
15 ÷ 2z	17	$\frac{zz - aa}{2z} = y$ } Having found the Value of a from the 24th Step, e and y will be easily found by these two Steps, and b u by the 9th and 10th Step.
16 ÷ 2s	18	$\frac{ss - aa}{2s} = e$ }
17 + 18	19	$\frac{zz - aa}{2z} + \frac{ss - aa}{2s} = y + e$



6, 19

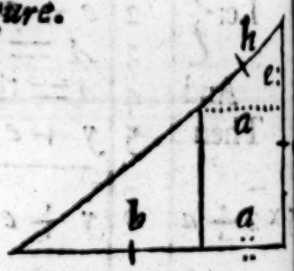
6, 19	20	$\frac{zz-aa}{2z}+\frac{ss-aa}{2s}=\frac{2A}{a}$
20 × 2z	21	$zz-aa+\frac{zss-zaa}{s}=\frac{4zA}{a}$
21 × s	22	$zzs-saa+zss-zaa=\frac{4zAs}{a}$
22 × a	23	$zzsa-saaa+zssa-zaaa=4zAs$
23, Num.	24	$900000a-aaa=243000000$

Here $a=300$ found as in the laſt *Problem.*

PROBLEM XIX.

There is a Right-angled Triangle, wherein a Right-line is drawn parallel to the Cathetus; *there is given the* Cathetus, *that Segment of the* Hypothenuſe *next to the* Cathetus, *and the alternate Segment of the* Baſe; *Thence to find the* Baſe, &c.

viz. Let	1	$b=20$. $c=24$. and $h=15$
Then	2	$b+a=$ the *Baſe*. Quere a
Here	3	$b+a:c::a:e$ Per *Figure.*
And	4	$aa+ee=hh$ Per *Figure.*
3 ∵	5	$\frac{ca}{b+a}=e$
5 ⊙ 2	6	$\frac{ccaa}{bb+2ba+aa}=ee$
4 − aa	7	$hh-aa=ee$
6, 7	8	$\frac{ccaa}{bb+2ba+aa}=hh-aa$
8 ×	9	$ccaa=hhbb-bbaa+2hhba-2ba^3+hhaa-a$
9 ±	10	$a^4+2baaa+ccaa+bbaa-hhaa-2hhba=hhb$
That is,	11	$aaaa+40aaa+751aa-9000a=90000$



For a Solution of this *Æquation*, let it be made

$aaaa+baaa+caa-da=C$ *Viz.* $\begin{cases} b=40 \text{ . } c=751 \\ d=9000 \text{ . } C=90000 \end{cases}$

Put $r+e=a$

Then $\left.\begin{cases} r+4rre+6rree=a^4 \\ brrr+3brre+3bree=baaa \\ crr+2cre+cee=caa \\ -dr-de=-da \end{cases}\right\}=C=90000$

Let $r=10$

Then

Then $\left\{\begin{array}{l} +10000 + 4000e + 600ee \\ +40000 + 12000e + 1200ee \\ +75100 + 15020e + 751ee \\ -90000 - 9000e \end{array}\right\} = C = 90000$

That is, $35100 + 22020e + 2551ee = 90000$

Hence it will be $22020e + 2551ee = 54900$

Consequently, $8,63e + ee = 21,52 = D$

And $\frac{D}{8,63 + e} = e$

Operation, 8,63) 21,52 (2,1 = *e*

$+e = 2,1$ — 20

Divisor	= 10	1,52	First $r = 10$
Divisor	= 10,7	1,07	$+e = 2,1$
		45 &c.	$r + e = 12,1 = r$ for a

second Operation, which being involv'd, and multiply'd into the *Coefficients*, as before, will produce these *Numbers*:

$$\left\{\begin{array}{l} +21435,8881 + 7086,24e + 878,46ee \\ +70862,4400 + 17569,20e + 1452,00ee \\ +109953,9100 + 18174,20e + 751,00ee \\ -108900,0000 - 9000,00e \end{array}\right\} = C$$

Viz. $93352,2381 + 33829,64e + 3081,46ee = 90000$

Here, because $93352,2381 > 90000$ Therefore $12,1 > a$, and therefore it must be made $r - e = a$, which will produce the same *Numbers*, only all the *second Sines* must be chang'd.

Thus, $93352,2381 - 33829,64e + 3081,46ee = 90000$ from whence will arise this *Æquation*,

$+33829,64e - 3081,46ee = 3352,2381$

Consequently, $10,9784e - ee = 1,08787332 = D$

Operation 10,9784) 1,08787332 (0,0999 = *e*

$-e = ,0999$ — 9792

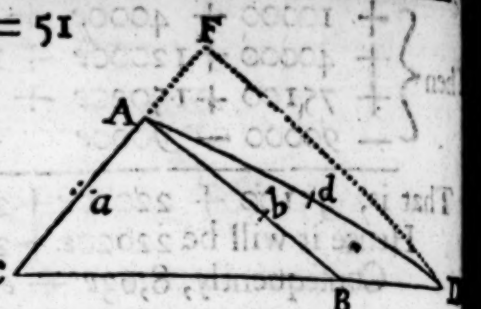
Divisor	10,88	108673	Last $r = 12,1$
Divisor	10,879	97911	$e - = 0,0999$
Divisor	10,8785	1076232	$r - e = 12,0001 = a$
		979065	
		&c.	

PROBLEM XX.

In the Oblique-angled Triangle CAD there is given the Side AD, and the Sum of the Sides AC+CD; also within the Triangle is given the Line AB Perpendicular to the Side CA; Thence to find the Side CA, &c.

Let

Let	1	$CA + CD = s = 51$
	2	$AD = d = 32$
	3	$AB = b = 21$
And	4	$CA = a$ fought
Then	5	$s - a = CD$



Suppoſe the *Line D F Parallel* to *A B*, and *C A* produc'd to *F*

Then $\triangle CAB$, and $\triangle CFD$ will be alike.

And	6	$BC : CA :: DC : CF$
But	7	$BC = \sqrt{bb + aa}$. Let $AF = e$, and $FD = y$
6, 7	8	$\sqrt{bb + aa} : a :: s - a : a + e$
8 ∴	9	$\frac{sa - aa}{\sqrt{bb + aa}} = a + e$
5 ⊙ 2	10	$ss - 2sa + aa = \square CD$
per *Fig.*	11	$ss - 2sa + aa = aa + 2ae + ee + yy = \square CF + \square FD$
11 — *aa*	12	$ss - 2sa = 2ae + ee + yy$
But	13	$dd = ee + yy = \square AF + \square FD$
12 — 13	14	$ss - 2sa - dd = 2ae$
Let	15	$2x = ss - dd$
14 15	16	$x - sa = ae$
16 ÷ *a*	17	$\frac{x - sa}{a} = e$
17 + *a*	18	$\frac{x - sa + aa}{a} = a + e$
9 ⊙ 2	19	$\frac{ssaa - 2saaa + a^4}{bb + aa} = \square \overline{a + e}$
18 ⊙ 2	20	$\frac{xx - 2xsa + 2xaa + ssaa - 2sa^3 + a^4}{aa} = \square \overline{a + e}$
18, 19	21	$\frac{ssaa - 2saaa + a^4}{bb + aa} = \frac{xx - 2xsa + 2xaa + ssaa - 2sa^3 + a^4}{aa}$

This *Æquation* being brought out of the *Fractions*, and into *Numbers*, will become

$$-2018aaaa + 125409 - 2464230{,}25aa + 35468307 = 274183922{,}25$$

which being divided by 2018, the *Co-efficient* of the highest *Power* of *a* will be $-a^4 + 62{,}1456a^3 - 1221{,}125aa + 17575{,}9697 = 135869{,}138875$, &c.

And

And from hence the *Value* of *a* may be found, as in the *Last Problem*, due Regard being had to the *Signs* of every *Term*.

This Work of *Reducing*, or preparing *Æquations* for a *Solution* by *Division*, hath always been taught both by *Ancient* and *Modern* Writers of *Algebra*, as a Work so necessary to be done, that they do not so much as give a Hint at the *Solution* of any *Adfected Æquation* without it.

Now it very often happens, that in *Dividing* all the *Terms* of an *Æquation*, some of their *Quotients* will not only run into a long *Series*, but also into imperfect *Fractions*, (as in this *Æquation* above) which renders the *Solution* both tedious and *Imperfect*.

To remedy that *Imperfection*, I shall here shew how this *Æquation* (and *Consequently* any other) may be *Resolv'd* without such *Division*, or *Reduction*.

Let $b = 2018$. $c = 125409$. $d = 2464230{,}25$
$f = 35468307$. And $G = 274183922{,}25$
Then the *precedent Æquation* will stand thus,

$$-baaaa + caaa - daa + fa = G$$

Put $r + e = a$ As before.

$$\text{Then will } \left\{\begin{array}{rrrr} -br^4 & -4brrre & -6brree & = -ba^4 \\ +cr^3 & +3crre & +3cree & = +ca^3 \\ -drr & -2dre & -dee & = -daa \\ +fr & +fe & \ldots\ldots & = +fa \end{array}\right\} = G$$

This is plain and easily conceiv'd; The next thing will be, how to Estimate the first *Value* of *r*; and to perform that, let *G* be *Divided* by *b*, only so far as to determine how many *Places* of whole *Numbers* there will be in the *Quotient*; Consequently, how many *Points* there must be (*according to the Height of the Æquation.*)

Thus $b = 2018)\ G = 274183922{,}25\ (130000$
2018
7238 &c.

Now from hence one may as easily guess at the *Value* of *r*, as if all the *Terms* had been *Divided*. That is, I suppose $r = 10$, which being *Involv'd*, &c. as the *Letters above direct*, will be

$$\left.\begin{array}{l} -\ 20180000 - 8072000e - 1210800ee \\ +\ 125409000 + 37622700e + 3762270ee \\ -\ 246423025 - 49284605e - 2464230,25ee \\ +\ 354683070 + 35468307e \end{array}\right\} = G$$

Viz. $213489045 + 15734402e + 87239,75ee = 2741839$ &c.

Hence $15734402e + 87239,75ee = 60694877,25$

Conſequently, $180,3e + ee = 695,72 = D$

And $\frac{D}{180,3 + e} = e$

Operation

$+e =$	3,7	549	
1. *Diviſor* $=$	183	146,72	First $r = 10$
2. *Diviſor* $=$	184,0	128,80	$+ e = 3,7$
		&c.	$r + e = 13,7 = r$ fo

180,3) 695,72 (3,7 $= e$

a ſecond *Operation*, with which you may proceed, as in th Laſt *Problem*, and ſo on to a Third *Operation*, if Occaſion re quire ſuch Exactneſs. But this may be ſufficient to ſhew th Method of Reſolving any *Adfected Æquation*, without reducin it; which is not only very exact, but alſo very ready in Pra ctice, as will fully appear in the laſt *Chapter* of this *Part*, con cerning the *Periphery* and *Area* of the *Circle*, &c. wherein yo will find a farther Improvement in the *Numerical Solution* o *High Æquations* than hath hitherto been publiſh'd.

CHAP. V.

Practical Problems, *and* Rules *for finding the* Superficial Contents, *or* Areas *of Right-lin'd Figures.*

BEfore I proceed to the following *Problems*, it may be con venient to acquaint the *Learner*, That the *Superfi ies* o *Area* of any *Figure*, whether it be *Right-lin'd* or *Circular*, *compos'd* or made up of *Squares*, either *Greater*, or *Leſs*, accord ing to the different *Meaſures* by which the *Dimenſions* of th *Figure* are taken or *Meaſur'd*.

That is, if the *Dimenſions* are taken in *Inches*, the *Area* wi be *compos'd* of *Square Inches*; if the *Dimenſions* are taken i *Feet*, the *Area* will be *compos'd* of *Square Feet*; if in *Yards*, th *Area* will be *Square Yards*; and if the *Dimenſions* are taken b *Poles* or *Perches*, (*as in Surveying of Land*, &c.) then the *Are* will be *Square Perches*, &c. Theſe Things being underſtood an

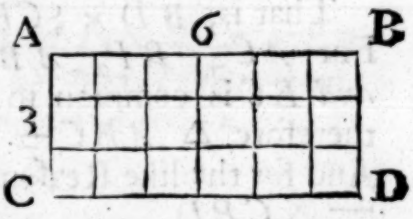
and the *Definitions* in the 283 and 284 *Pages* well consider'd, will help to render the following *Rules* very easy.

PROBLEM I.

To find the Superficial Content, or Area of a Square ; *or of any Right-angled* Parallelogram.

Rule { *Multiply the Length into its Breadth, and the Product will be the Area requir'd.* (*See* Lemma 1. Pag. 302.)

Example, Suppose the *Line AB* = 6 *Yards*, and the *Breadth AC* or *BD* = 3 *Yards*.



Then $AB \times AC = 6 \times 3 = 18$ will be the Number of *Square Yards* contain'd in the *Area* of the *Parallelogram ABCD*.

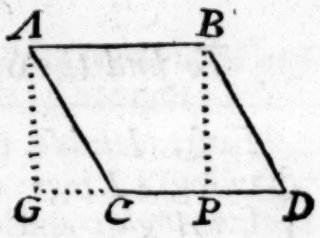
This is so evident by the *Figure* only, that it needs no *Demonstration*.

PROBLEM II.

To find the Area of any Oblique-angled Parallelogram, viz. *either of a* Rhombus *or* Rhomboides.

Rule { *Multiply the Length into its perpendicular Height*, (*or Breadth*) *and the Product will be the Area requir'd.*

That is, the Side $AB \times BP =$ the *Area* of the *Rhombus ABCD*. For if *BP* be drawn *perpendicular* to *CD*, and *AG* be made *parallel* to *BP*, then will $GC = PD$ and $GP = CD$. Consequently $\triangle AGC = \triangle BPD$, and $\square ABGP =$ *Rhombus ABCD*.



But $AB \times BP = \square ABGP$. Therefore $AB \times BP$, or $CD \times BP =$ the *Area* of the *Rhombus ABCD*.

Example, Suppose the Side *AB* = 23 *Inches*, and the *Perpendicular* BP = 17,5 *Inches*, (*being the shortest or neareſt Diſtance between the two Sides AB, and CD.*)

Then $AB \times BP = 23 \times 17{,}5 = 402{,}5$ *Square Inches*, being the *Area* of the *Rhombus* requir'd.

The like may be done for any *Rhomboides* whose *Length* and *Perpendicular Breadth* are given.

PROBLEM III.

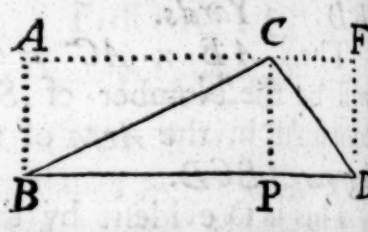
To find the Superficial Content, or Area of any plain Triangle.

Every plain Triangle is Equal to half its Circumscribing Parallelogram, (41. *e.* 1.) *which affords the following Rule.*

Rule { *Multiply the Base of the given Triangle into Half its perpendicular Height, Or Half the Base into the whole Perpendicular, and the Product will be the Area.*

That is, $BD \times \frac{1}{2}CP$, or $\frac{1}{2}BD \times CP =$ *Area* of $\triangle BCD$.

For $AC = BP$, $AB = CP$, *and* BC is common to both $\triangle\triangle$ therefore $\triangle ABC = \triangle BCP$. And for the like Reasons $\triangle CFD = \triangle CPD$.



Therefore $\triangle BCP + \triangle CPD = \frac{1}{2} \square ABCD$. Consequently $\frac{1}{2}BD \times CP$, or $BD \times \frac{1}{2}CP$ will be the *Area* of the $\triangle BCD$.

Example, Suppose the *Base* $BD = 32$ *Inches*, and the *perpendicular Height* $CP = 14$ *Inches*.

Then $\frac{1}{2}BD \times CP = 16 \times 14 = 224$. Or $BD \times \frac{1}{2}CP = 32 \times 7 = 224$.

Or thus, $32 \times 14 = 448$. Then $2)448(224 =$ the *Area* of the *Triangle BCD* in *Square Inches*.

PROBLEM IV.

To find the Superficies, or Area of any Trapezium.

First, *Divide* the given *Trapezium* into *Two Triangles*, by drawing a *Diagonal* from one of its Acute *Angles* to the Opposite *Angle*; and let fall *Two Perpendiculars (from the other Two Angles)* upon the *Diagonal*. As in the following *Figure*. Then

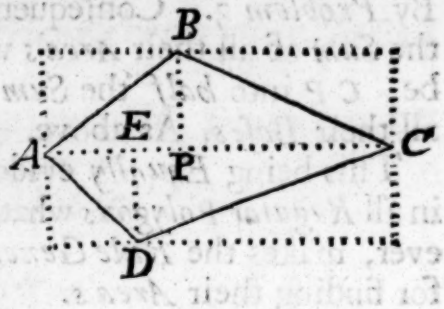
Rule { *Multiply half the Diagonal into the Sum of the Two Perpendiculars, or half the Sum of the Perpendiculars into the Diagonal, and the Product will be the Area.*

That is, $\frac{1}{2}AC \times \overline{BP + ED}$. Or $AC \times \frac{1}{2}BP + \frac{1}{2}ED =$ *Area* of the *Trapezium ABCD*.

For the $\triangle ABC$ is *Half* its *Circumscribing Parallelogram*. And the $\triangle ACD$ is also Half of its *Circumscribing Parallelogram*, As hath been prov'd at the last *Problem*.

Consequent

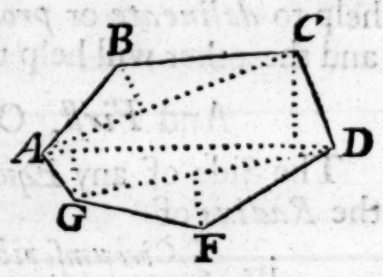
Conſequently, $BP + ED \times \frac{1}{2} AC$, or $\frac{1}{2} BP + \frac{1}{2} ED \times AC$ will be the *Area* of the *Trapezium*, As above.



Example, Suppoſe the *Diagonal* $AC = 33$ *Feet*, the *Perpendicular* $BP = 15$ *Feet*, and the *Perpendicular* $ED = 14$ *Feet*. Then $BP + ED = 29$ *Feet* and $BP + ED \times \frac{1}{2} AC = 29 \times 16{,}5 = 478{,}5$. Or $AC \times \frac{1}{2} BP + \frac{1}{2} ED = 33 \times \frac{29}{2} = 478{,}5$. Or thus, $29 \times 33 = 957$. Then 2) 957 (478,5 any of theſe *Products* are the *Area* of the *Trapezium ABCD*.

PROBLEM V.

To find the Superficial Content or Area of any Irregular Polygon or many Sided Figure, which by ſome Authors is call'd a Triangulate, becauſe (as I ſuppoſe) it muſt be Divided into Triangles, as in the *Annexed Figure ABCDFG*; by which it is evident, that the *Sum* of the *Area's* of all thoſe *Triangles*, found as in the laſt *Problem*, &c. will be the *Area* of their *Circumſcribing Polygon*.



PROBLEM VI.

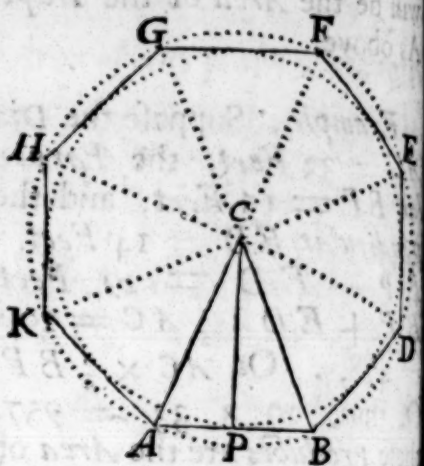
To find the Superficies, or Area of any Regular Polygon, viz. *of any Regular* **Pentagon, Hexagon, Heptagon, Octagon,** *&c.*

General **Rule** { *Multiply half the Sum of its Sides into the Radius of the Inſcrib'd Circle, Or half the ſaid Radius into the Sum of the Sides, and the Product will be the Area requir'd.*

That is, $\frac{AB+BD+DE+EF+FG+GH+HK+KA}{2} \times CP$ = the *Area* of the *Annexed Octagon*: wherein it is evident, that its *Area* is compos'd of ſo many *Equal Iſoſceles Triangles* as there are *Number* of *Sides in the Polygon*, viz. of *Eight Iſoſceles Triangles*, whoſe *Baſes* are the *Sides* of the *Octagon*, viz. $AB = BD = DE$, &c. And the *Sides* of thoſe *Triangles*, CA, CB, CD, &c. are the *Radius's* of the *Circumſcribing Circle*; and their *perpendicular Heights*, viz. CP, is the *Radius* of the *Inſcrib'd Circle*.

But

But the *Area* of any one of those *Triangles*, is $\frac{1}{2}$ *A B* × *C P*. By *Problem* 3. Consequently the *Sum* of all their *Area's* will be, *C P* into *half* the *Sum* of all their *Bases*, As above.



This being *Equally* evident in all *Regular Polygons* whatsoever, makes the *Rule General* for finding their *Area's*.

Now becauſe it is requir'd to have the *Radius* of the propos'd *Polygon's Inſcrib'd Circle*, I ſhall here inſert (*and demonſtrate*) the *Proportions* that are between the *Sides* of ſeveral *Regular Polygons*, and the *Radius* both of their *Inſcrib'd* and *Circumſcribing Circles*: the one will help to *delineate* or *project* the *Polygon*, (*if Occaſion require it*) and the other will help to find its *Area*.

And *Firſt*, Of an Equilateral Triangle.

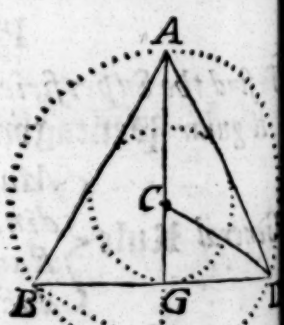
The Side of any *Equilateral plain Triangle* is in *Proportion* to the *Radius* of

its { *Circumſcribing Circle*, As 1 : To 0,57735027 &c.
{ *Inſcrib'd Circle*, As 1 : To 0,28867513 &c.

And to its *Perpendicular Height*, As 1 : To 0,86602540 &c.

That is, { *A B* : *C D* :: 1 : 0,57735027
{ *A B* : *C G* :: 1 : 0,28867513

And *A B* : *A G* :: 1 : 0,86602540



Demonſtration.

Let *A B* = *B D* = 1
Then will *B C* = *G D* = 0,5
But □ *A B* − □ *B G* = □ *A G*
By *Theorem* 11.
That is, 1 − 0,25 = 0,75 = □ *A G*.
Conſequently, √ 0,75 = 0,86602540 = *A G*:
Then *A G* : *A B* :: *A B* :: *A H*. By *Theorem* 13.
That is, 0,8660254 : 1 :: 1,15470054 &c. = *A H*
Then $\frac{1}{2}$ *A H* = 0,57735027 = *A C*. Again *A G* : *D G* : *D G* : *C*
That is, 0,8660254 : 0,5 :: 0,28867513 = *C G*. Q. E.

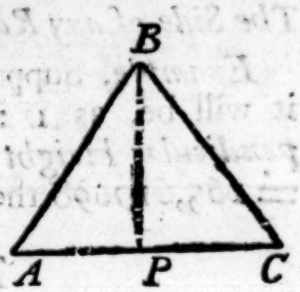
Now, by the Help of the *Firſt* of theſe *Proportions*, it will be eaſy to *Reſolve* the following *Problem*.

PROB

PROBLEM VII.

The Side of any Equilateral plain Triangle being given, To find its Area.

Example, Suppose the Side of the propos'd *Triangle ABC* be 25 *Inches*, viz. $AB = BC = CA = 25$



First 1 : 0,866254 :: $AB = 25$: 21,650635 $= BP$. By *Theorem* 13.

Then $AP\,(= \frac{1}{2}CA) \times BP =$ the *Area* of $\triangle ABC$. By *Rule* to *Problem* 3.

That is, 12,5 x 21,650635 = 270,6329 the *Area* in *Square Inches*.

Or this *Problem* may be otherwise resolv'd, thus:

Let $b = AP = \frac{1}{2}AC$. Then $2b = AB$.

But $\square AB - \square AP = \square BP$. By *Theorem* 11.

That is, $4bb - bb = 3bb = \square BP$. Consequently, $\sqrt{3bb} = BP$

Then $b\sqrt{3bb} = BP \times \frac{1}{2}AC$. Viz. $\sqrt{3bbbb}$ = the *Area* of the *Triangle*.

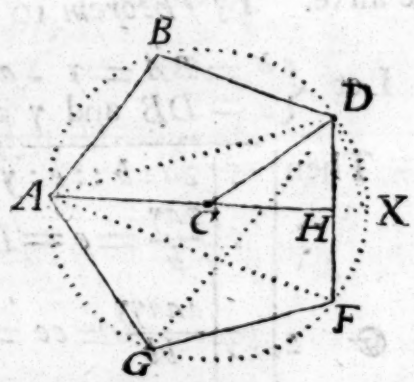
Secondly, For a Pentagon.

The Side of any Pentagon, is in Proportion to the Radius of

its { *Circumscribing Circle*, As 1 : To 0,85065080 &c.
{ *Inscrib'd Circle*, As 1 : To 0,68819096 &c.

and to its *Perpendicular Height*, As 1 : To 1,53884176 &c.

Viz. { $AB : AC :: 1 : 0,85065080$
{ $AB : CH :: 1 : 0,68819096$
{ $AB : AH :: 1 : 1,53884176$



Demonstration.

Let $AB = 1$. And draw the Diagonals *AD*, *AF* and *DG*, which will be *Equal* to one another. Then will $AG \times DF + AD \times GF = AF \times DG$ by *Theorem* 19.

Consequently, $AG \times DF = AF \times DG : - AD \times GF$

That is, $\square AB = \square AD : - AD \times GF = 1$

Hence it will be $AD = 1,61803398$

Then $\square AD - \square DH = \square AH$ By *Theor.* 11. But $DH = \frac{1}{2}AB$

Therefore $\sqrt{\square AD - \frac{1}{4}\square AB} = AH = 1,53884176$.

Again, $AH : AD :: AD : AX = 2AC$. For $\triangle AHD$ and $\triangle ADX$ are alike.

Ergo

Ergo $\frac{\square AD}{AH} = 2AC = 1{,}70130161$. Hence $AC = 0{,}85065080$

But $AH - AC = CH = 0{,}68819096$ &c. Q. E. D.

From hence it will be *Easy* to *Resolve* the following *Problem*.

PROBLEM VIII.

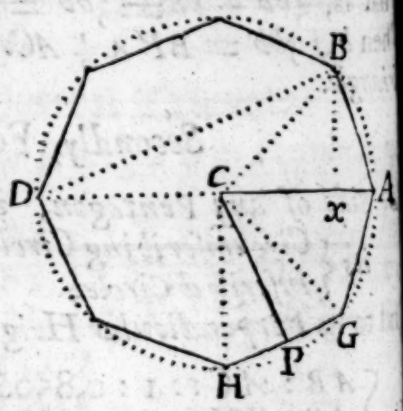
The Side of any Regular Pentagon being given, To find its Area.

Example, Suppose the given *Side* to be 15 *Inches* long, then it will be, as 1 : 1,53884176 : : 15 : 22,0826264 the *perpendicular Height*; and by the general *Rule* $22{,}0826264 \times \frac{4}{2} = 165{,}619698$ the *Area* requir'd.

Thirdly, For an Octagon.

The Side of any Regular Octagon is in Proportion to the Radius of its { *Circumscribing Circle*, As 1 : To 1,30656296 &c. / *Inscrib'd Circle*, As 1 : To 1,20710678 &c.

Viz. { $BA : CA :: 1 : 1{,}30656296$ / $BA : CP :: 1 : 1{,}20710678$



Demonstration.

Draw the *Right Line DB*, and from the *Point B* let fall the *Perpendicular B x* upon the *Diameter DA*.

Then will $\triangle DBA$ and $\triangle DxB$ be alike. By *Theorem* 10 and 12.

Let $\begin{cases} b = BA = 1 \text{ . } a = CA \\ e = DB \text{ and } y = Bx \end{cases}$

Then	1	$2a : b :: e : y$ Viz. $DA : BA :: DB : Bx$
1 ∴	2	$\frac{2ay}{b} = e = DB$
2 ⊙ 2	3	$\frac{4aayy}{bb} = ee = \square DB$
But	4	$4aa - \frac{4aayy}{bb} = bb$
That is		$\square DA - \square DB = \square BA$. By *Theorem* 11.
$4 \times bb$	5	$4bbaa - 4aayy = bbbb$
Again	6	$\begin{cases} \frac{1}{2}aa = yy. \text{ For } Cx = Bx \\ \text{and } \square Cx + \square Bx = \square CB = aa \end{cases}$

5,

, 6	7	$4bbaa - 2a^4 = b^4$ Or $2a^4 - 4bbaa = -b^4$
÷ 2	8	$aaaa - 2bbaa = -\frac{1}{2}b^4$
C[]	9	$a^4 - 2bbaa + b^4 = b^4 - {}^1b^4 = \frac{1}{2}b^4$
ш 2	10	$aa - bb = \sqrt{\frac{1}{2}b^4}$
o+bb	11	$aa = bb + \sqrt{\frac{1}{2}b^4}$
1 ш 2	12	$a = \sqrt{:bb + \sqrt{\frac{1}{2}b^4}} = 1,30656296$, &c. $= CA$
Then	13	$aa - \frac{1}{4}bb = \square CP$, viz. $\square CH - \square HP = []CP$
3 ш 2	14	$\sqrt{aa - \frac{1}{4}bb} = 1,20710678$ &c. $= CP$.

From hence 'twill be easie to find the *Area* of any *Octagon*.

PROBLEM IX.

The Side of any Regular Octagon *being given, To find its* Area.

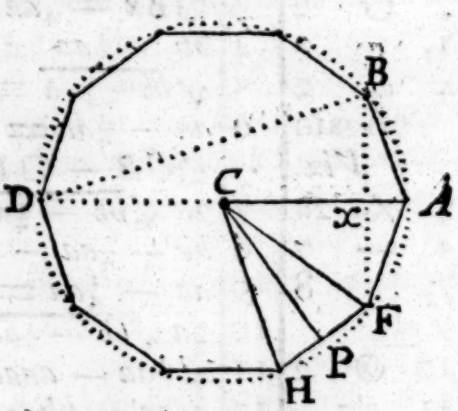
Example. Suppose the Side given to be 12 inches long; *First*, s 1 : 1,20710678 : : 12 : 14,48528136 = the *Radius* of its in-ib'd Circle; then 12 × 4 = 48 is half the Sum of its Sides, nd 48 × 14,48528136 = 695,2935 the *Area* requir'd.

Fourthly, For a Decagon.

The Side of any Regular Decagon (viz. *a* Polygon *of Ten equal Sides*) *is in Proportion to the* Radius *of*

Its { *Circumscribing Circle*, as 1 : to 1,61803398 &c.
{ *Inscrib'd Circle*, as 1 : to 1,53884176 &c.

viz. { $BA : CA :: 1 : 1,61803398$
{ $BA : CP :: 1 : 1,53884166$



Demonstration.

Let { $b = BA = 1$. $a = CA$
{ $c = DB$. and $y = Bx$

Then	1	$2a : b :: c : y$
That is,		$DA : BA :: DB : Bx$
::	2	{ $2ay = bc$ and $2y = \frac{bc}{a}$.
But	3	$2y : c :: 1 : 1,61803398$ See *Pentagon*.
∴	4	$\frac{1c}{1,61803398} = 2y = \frac{bc}{a} = \frac{1c}{a}$
÷ 1c	5	$1,61803398 = a = CA$
Again	6	{ $aa - \frac{1}{4}bb = \square CP$. viz. $\square CF - \square PF = \square CP$. By *Theorem* 11.
That is,	7	$\sqrt{2,01803396 = 0,25} = 1,53884176 = CP$

PROBLEM X.

The Side of any Regular Decagon *being given, To find its* Area.

Example. Let the given Side be 14 inches long; then as 1 : 1,53884176 : : 14 : 21,543784 = the *Radius* of the *inscrib'd Circle*; and 14 × 5 = 70 is half the Sum of its Sides. Lastly, 21,543784 × 70 = 1508,06488 the *Area* requir'd.

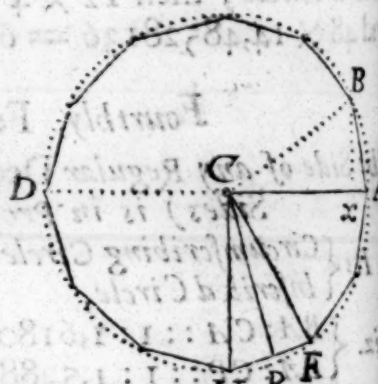
Fifthly, For a Dodecagon.

The Side of any Regular Dodecagon (viz. *a* Polygon *of Twelve equal Sides*) *is in Proportion to the* Radius *of*

Its { *Circumscribing Circle*, as 1 : to 1,93185165 &c.
Its { *Inscrib'd Circle*, as 1 : to 1,86632012 &c.

Viz. { $BA : CA :: 1 : 1{,}93185165$
Viz. { $BA : CP :: 1 : 1{,}86632012$

Demonstration.



Let $b = BA = 1$. $a = CA$ as before
And $e = xA$; then $a - e = Cx$

	First	1	$bb - \square Bx = ee$ by Figure.
	But	2	$Bx = \frac{1}{2}CA = \frac{1}{2}a$
2	⊙ 2	3	$\square Bx = \frac{1}{4}aa$
1,	3	4	$bb - \frac{1}{4}aa = ee$
4	ww 2	5	$\sqrt{bb - \frac{1}{4}aa} = e$
	Again	6	$aa - \frac{1}{4}aa = aa - 2ae + ee$
	Viz.		$\square CB - \square Bx = \square Cx$
5	× 2a	7	$2a\sqrt{bb - \frac{1}{4}aa} = 2ae$
4	— 7	8	$bb - \frac{1}{4}aa - 2a\sqrt{bb - \frac{1}{4}aa} = ee - 2ae$
7,	8	9	$aa - \frac{1}{4}aa = aa + bb - \frac{1}{4}aa - 2a\sqrt{bb - \frac{1}{4}aa}$
9	+	10	$2a\sqrt{bb - \frac{1}{4}aa} = bb$
10	⊙ 2	11	$4bbaa - aaaa = b^4$
11	+	12	$aaaa - bbaa = -b^4$
13	C□	13	$aaaa - 4bbaa + 4b^4 = 3b^4 = 3$
1	ww 2	14	$aa - 2bb = \sqrt{3} = 1{,}7320508075$
14	+ 2bb	15	$aa = 2bb + \sqrt{3} = 3{,}7320508075$
15	ww 2	16	$a = \sqrt{3{,}7320508075} = 1{,}93185165 = CA$
	Again	17	$aa - \frac{1}{4}bb = \square CP$. viz. $\square CF - \square PF = \square C$
17,	Hence	18	$CP = \sqrt{aa - \frac{1}{4}bb} = 1{,}86632012$ Q. E.

Co

Conſectary.

Hence, if the Side of any *Regular Dodecagon* be given, the *Radius* of its *Inſcrib'd Circle* may be eaſily obtain'd, and thence the *Area* found; as in the laſt Problem.

The Work of the foregoing *Polygons*, being well conſider'd, will help the young *Geometer* to raiſe the like Proportions for others, if his Curioſity requires them: And not only ſo, but they will alſo help to form a true Idea of a Circle's *Periphery* and *Area*, according to the Method which I ſhall lay down in the next Chapter for finding them both.

CHAP. VI.

A New and Eaſie Method of finding the Circle's Periphery *and* Area *to any aſſign'd Exactneſs (or Number of Figures) by one Æquation only. Alſo a new and facile Way of making Natural* Sines *and* Tangents.

LET us ſuppoſe *(what is very eaſie to conceive)* the Circle's *Area* to be compos'd or made up of a vaſt number of *plain Iſoſceles Triangles*, having their *Acuteſt Angles* all meeting in the Circle's Centre. And let us imagine the *Baſes* of thoſe *Triangles* ſo very ſmall, that their Sides and their Perpendicular Heights, *viz.* the *Radius's* of their Circumſcrib'd and Inſcrib'd Circles (*vide* Problem 6.) may become ſo very near in *Length* to each other, as that they may be taken one for another without any ſenſible Error: Then will the *Peripheries* of their Circumſcribing and Inſcrib'd Circles become *(altho' not co-incident, yet)* ſo very near to each other, as that either of them may be indifferently taken for one and the ſame Circle.

But how to find out the Sides of a *Polygon* (viz. *the Baſes of thoſe Iſoſceles Triangles)* to ſuch a convenient Smallneſs as may be neceſſary to determine and ſettle the Proportion betwixt a Circle's *Diameter* and its *Periphery*, *(to any aſſign'd Exactneſs)* hath hitherto been a Work which requir'd great *Care* and much *Time* in its Performance; as may eaſily be conceiv'd from the *Nature* of the *Method* us'd by all thoſe who have made any conſiderable Progreſs in it, viz. *Archimedes*, *Snellius*, *Hugenius*, *Mætius*, *Van Culen*, &c. Theſe proceeded with the Biſecting of an *Arch*, and found the *Value* of its *Chord*, to a convenient number of Fi-

gures at every ſingle Biſection, repeating their *Operations* until they had approach'd to the *Chord* deſign'd.

And this *Method* is made choice of by the Learned Dr. *Wallis* in his Treatiſe of *Algebra*; wherein, after he hath given us a large Account of the different Enquiries made by ſeveral, *(very eminent in Mathematical Sciences)* in order to find out ſome eaſier and more expeditious Way of approaching to the Circle's *Periphery*, as in *Chap.* 82, 84, 85, 86, and ſeveral other places, he comes to this Reſult, *(page* 321.)

" 'Tis true, *ſaith he*, we might in like Manner proceed by con-
" tinual Triſection, Quinquiſection, or other Section, if we had
" for theſe as convenient Methods of Operation as we have for
" Biſection: But becauſe *Euclid* ſhews how to biſect an Arch Geo-
" metrically, but not to triſect, *&c.* and the one may be done
" *(Algebraically)* by reſolving a Quadratick Æquation, but not
" thoſe other, without Æquations of a higher compoſition, I there-
" fore make choice of a continual Biſection, *&c.*

And then he lays down theſe following Canons.

The Subtenſe of $\frac{1}{4}$	1	into 6
of $\frac{1}{12}$	$\sqrt{:2-\sqrt{3}}$	into 12
of $\frac{1}{24}$	$\sqrt{:2-\sqrt{:2+\sqrt{3}}}$	&c. 24
of $\frac{1}{48}$	$\sqrt{:2-\sqrt{:2+\sqrt{:2+\sqrt{3}}}}$	48
of $\frac{1}{96}$	$\sqrt{:2-\sqrt{:2+\sqrt{:2+\sqrt{:2+\sqrt{3}}}}}$	96
&c.	$\sqrt{:2-\sqrt{:2+\sqrt{:2+\sqrt{:2+\sqrt{:2+\sqrt{3}}}}}}$	102
	$\sqrt{:2-\sqrt{:2+\sqrt{:2+\sqrt{:2+\sqrt{:2+\sqrt{:2+\sqrt{3}}}}}}}$	384
	$\sqrt{:2-\sqrt{:2+\sqrt{:2+\sqrt{:2+\sqrt{:2+\sqrt{:2+\sqrt{:2+\sqrt{3}}}}}}}}$	768
	&c.	&c.

How tedious and troubleſome the Work of theſe complicated Extractions is, I leave to the Conſideration of thoſe, who either have had Experience therein, or out of Curioſity will give themſelves the Trouble of making Tryal.

Again, in *page* 347, the *Doctor* inſerts a particular Method propos'd by *Libnitius*, publiſh'd in the *Acta Eruditorum* at *Leipſick*, for the Month of *February* 1682, in order to find the Circle's *Area*, and conſequently its *Periphery*, which is this:

As 1 : to $\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\frac{1}{17}-\frac{1}{19}$ &c. Infinitely : : ſo is the Square of the *Diameter* : to the Circle's *Area*. But this convergeth ſo very ſlowly, that it is not worth the Time to purſue it.

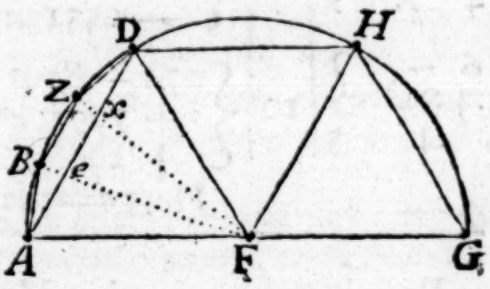
I ſhall here propoſe a New Method of my own, whereby the Circle's *Periphery*, and conſequently its *Area*, may be obtain'd infinite-

Infinitely near the Truth, with much greater Eaſe and Expedition than either that of *Biſection*, or that of *Libnitius*, as above, or any other Method that I have yet ſeen, it being perform'd by *Reſolving* only one *Æquation*, deduced by an Eaſie Proceſs from the Property of a Circle *(known to every Cooper)* which is this:

The Radius *of every Circle is Equal to the* Chord *of One ſixth part of its* Periphery.

That is, $AD = DH = HG$, the *Chords* of One third part of the *Semicircle*, are each Equal to AF its *Radius*.

Then, if the Arch AD be Triſected, it will be $AB = BZ = ZD$.



Let $\begin{cases} R = AF = 1 \\ c = AD = 1 \\ a = AB \end{cases}$ Quere a.

Then	1	$R : a :: a : \frac{aa}{R} = Be$
And	2	$R : a :: R : -\frac{aa}{R}c - 2a$
That is, For	3	$FB : BZ :: Fe : ex = AD - 2a$ $\triangle AFB$, and $\triangle BAe$, are alike. And $AB = Ac = Dx$, &c.
∴	4	$Re - 2Ra = Ra - \frac{aaa}{R}$
X &c.	5	$3Ra - aaa = RRe$. That is, $3a - aaa = 1$. Here a = the Chord of $\frac{1}{18}$ part of the *Circle*. For $\frac{1}{6}$ or $\frac{1}{6} = \frac{1}{18}$

Next, *To triſect the* Arch AB.

Let	1	$3y - y^3 = a$ the laſt Chord.
@ 3	2	$27y^3 - 27y^5 + 9y^7 - y^9 = a^3$
X 3	3	$9y - 3y^3 = 3a$
− 2	4	$9y - 30y^3 + 27y^5 - 9y^7 + y^9 = 3a - a^3 = 1$ Here y = the Chord of $\frac{1}{54}$ part of the Circle.

Again, *To triſect the* Arch *whereof* y *is the* Chord.

Let	1	$3a - a^3 = y$
@ 3	2	$27a^3 - 27a^5 + 9a^7 - a^9 = y^3$
@ 5	3	$243a^5 - 405a^7 + 270a^9 - 90a^{11} + 15a^{13} - a^{15} - y^5$

1 @ 7.

1 ⊙ 7	4	$\left\{\begin{array}{l} 2187a^7 - 5103a^9 + 5103a^{11} - 2835a^{13} + \\ \quad + 945a^{15} \quad - y^7 \end{array}\right.$	
1 ⊙ 9	5	$\left\{\begin{array}{l} 19683a^9 - 59049a^{11} + 78732a^{13} - \\ \quad - 61236a^{15} = y^9 \end{array}\right.$	
1 × 9	6	$27a - 9a^3 = 9y$	
2 × 30	7	$810a^3 - 810a^5 + 270a^7 - 30a^9 = 30y^3$	
3 × 27	8	$\left\{\begin{array}{l} 6561a^5 - 10935a^7 + 7290a^9 + 2430a^{11} + \\ \quad + 405a^{13} + 27a^{15} = 27y^5 \end{array}\right.$	
4 × 9	9	$\left\{\begin{array}{l} 19683a^7 - 45927a^9 + 45927a^{11} - \\ \quad - 25515a^{13} + 8505a^{15} = 6y^7 \end{array}\right.$	
6 − 7 + 8 − 9 + 5	10	$\left\{\begin{array}{l} 27a - 819^3 + 7371a^5 - 30888a^7 + \\ \quad + 72930a^9 - 107406^{11} + \\ \quad + 104652a^{13} - 69768a^{15} \end{array}\right\} =$	

Here a = the Chord of $\frac{1}{168}$ part of the Circle.

Proceeding on in this Method of continually Triſecting the Arch of every new Chord, and ſtill connecting the produc'd *Æquations* into one, as in the *two laſt Triſections*, 'twill not be difficult to obtain the *Chord* of any aſſign'd Arch, how ſmall ſoever it be.

Now, in order to facilitate the Work of raiſing theſe *Æquations* to any conſiderable Height, 'twill be convenient to add a few uſeful Obſervations concerning the Nature, and of ſuch Contractions as may be ſafely made in them; which, being well underſtood, will render the Work very *eaſie*.

1. *I have obſerv'd, that every* Triſection *will gain or advance one Figure in the Circle's* Periphery, *but no more. Therefore ſo many places of Figures as are at firſt deſign'd to be perfect in the* Periphery, *ſo many* Triſections *muſt be repeated to raiſe an Æquation that will produce a* Chord *anſwerable to that Deſign.*

2. *I have alſo found, that all the Superiour Powers (of x) whoſe Indices are greater than the Number of* Triſections (*viz. whoſe Indices are greater than the Number of deſign'd Figures*) *may be wholly rejected as inſignificant.*

3. *When once the Number of* Triſections, *and thence the higheſt Power (of x) is determin'd, the Third* Proceſs (*viz. the Third* Triſection) *may be made a fix'd or conſtant Canon; for by it, and Multiplication only, all the ſucceeding* Triſections (*how many ſoever they are*) *may be compleated without repeating the ſeveral Involutions.*

4. I

4. *In raising and collecting the Co-efficients of the several Powers (of* a*) 'twill be sufficient to retain only so many significant Figures (at* a^3*) as there is design'd to be places of Figures in the* Periphery, *(or at most but two more) and every succeeding superiour power may be allow'd to decrease two places of significant Figures: But herein great care must be taken to supply the places of those Figures that are omitted with Cyphers, that so the whole and exact number of places may be truly adjusted, otherwise all the Work will be erroneous.*

Now the Number of those supplying Cyphers may be very conveniently denoted by Figures placed within a Parenthesis, *thus;* 576 (8) a^3, *may signifie* $57600000000a^3$, *as in the following Æquations. The like may be done with* Decimal Parts, *thus;* (7) 658 *may signifie* ,0000000658 &c. *which will be found very useful in the Solution of these and the like Æquations.*

The aforesaid Contractions may be safely made, because both the superiour Powers of *a*, which are rejected; as also those Numbers that are omitted in the *Co-efficients* (and supply'd with Cyphers) would produce Figures so very remote from Unity, as that they would not affect the *Chord* design'd; that is, they would not affect the *Chord* in that place wherein the design'd *Periphery* is concern'd; as will in part appear in the following Example.

If these Directions be carefully minded, 'twill be easie to raise an *Æquation* that will produce the Side of a *Regular Polygon*, whose number of Sides shall be vastly numerous, consequently infinitely small: But, I presume, 'twill be sufficient for an *Example* to find the Side of a *Polygon* consisting of 258280326 Equal Sides; that is, if I find the *Chord* of $\frac{1}{258280326}$ part of the *Circle*'s Periphery, and that requires but *Sixteen Trisections*, which being order'd as before directed, will produce this *Æquation*.

$$\left\{\begin{array}{l} 43046721a - 3323601794869686812(4)a^3 \\ +769837653199714(20)a^5 - 8491218532841(35)a^7 \\ +54633331143(50)a^9 - 230083348(66)a^{11} \\ +6830988(79)a^{13} - 15072(94)a^{15} \end{array}\right\} = 1$$

Here the Value of *a* will have 23 Places of *Figures* true; that is, the Sides of the *Inscrib'd* and *Circumscrib'd Polygons* will be exactly the same to 23 Places of *Decimal Parts*, but not farther, all which may be easily obtain'd at *Two Operations*. And for the first, 'twill be sufficient to take only *Three Terms* of the *Æquation*, which will admit of being yet farther contracted, thus:

Let

Let $\left\{\begin{matrix} 43046721a - 3323601794(12)a^3 \\ +76983765(27)a^5 \end{matrix}\right\} = 1$

And let $r + e = a$; then rejecting all the *Powers* of e, that ariſe by Involution above eee,

it will be $r^3 + 3rre + 3ree + eee = aaa$

And $r^5 + 5r^4e + 10r^3ee + 10rreee = a^5$

Then the firſt ſingle Value of r may be thus found:

43046721) 1,00000000 (,00000002 $= r$

This ,00000002 $= r$ being duly involv'd, and its Powers multiply'd into their reſpective *Co-efficients*, will produce

$$\left.\begin{matrix} +,86093441 + 43046721e \\ -,02658881 - 3988322e - 199416(9)ee - 3324(18)eee \\ +,00024635 + 61587e + 6159(9)ee + 308(18)eee \end{matrix}\right\} = 1$$

viz. ,83459196 + 39119986e − 193257(9)ee − 3016(18)eee = 1

Hence 39119986e − 193257(9)ee − 3016(18)eee = 0,16540804

All the *Terms* of this laſt *Æquation* being divided by 193257(9) the *Co-efficient* of ee, it will then become

,00000020 24e − ee − 156(5)eee = ,00000000000000008558968 = D

Conſequently, $\left\{\dfrac{D + 156(5)eee}{,00000020 24 - e} = e\right.$

Operation.

,00000020 24) ,00000000000000008558968 (,00000004 = e

$-e$,00000000 43 + ,00000000000000009984 = 156(5)eee

1 *Di.* ,00000019 8) ,00000000000000008568952 (00000004327

2 *Di.* ,00000019 81

7921

6489

5943

5465

3962

&c.

First r = ,00000002

$+e$ = ,00000004327

$r + e$ = ,00000024327 = a.

Or rather new r for a ſecond Operation.

Now, if this firſt Value of a = ,00000024327 *were not continued to more places of Figures by a ſecond Operation, but only multiply'd into the Number of* Chords,

Viz. ,00000024327 × 258280326 = 6,28318539 &c. *the Periphery of that* Circle *whoſe Diameter is* 2, *nearer than either*

Archimedes

Archimedes, *or* Mœtius's *Proportion: For* Archimedes *makes it* 6,285714 &c. *viz. As* 7 *To* 22. *And* Mœtius *makes it* 6,28318584 &c. viz. *As* 113 *To* 355.

But if the whole *Æquation* before propos'd be now taken, and we proceed to a Second *Operation*, the *Value* of *a* may be *encreas'd* with *Twelve* Places of *Figures* more, and those may be obtain'd by plain *Division* only.

Thus, let $r + e = a$, as before, and let all the *Powers* of e be now *Rejected* as insignificant;

Then will $\left\{\begin{array}{l} r + e = a \\ r^3 + 3r^2e = a^3 \\ r^5 + 5r^4e = a^5 \\ r^7 + 7r^6e = a^7 \end{array}\right\}$ and $\left\{\begin{array}{l} r^9 + 9r^8e = a^9 \\ r^{11} + 11r^{10}e = a^{11} \\ r^{13} + 13r^{12}e = a^{13} \\ r^{15} + 15r^{14}e = a^{15} \end{array}\right\}$

The several Powers of $r = ,000000024327$ being *Rais'd*, and *Multiply'd* into their respective *Co-efficients*, will produce these following *Numbers*.

+ 1,047197581767	+ 43046721*e*	
− ,047849196598394865	− 5900751*e*	
+ ,000655906484595355	+ 134810*e*	
− ,000004281440413375	− 1232*e*	= 1
+ ,000000016302517863	+ 6*e*	
− ,000000000040631167	− 0*e*	
+ ,000000000000071388	+ 0*e*	
− ,000000000000000093	− 0*e*	

Viz. 1,000000026474745106 + 37279554*e* = 1

Hence $37279554e = -,000000026474745106 = D$

Or rather $-37279554e = ,000000026474745106 = D$

Consequently, $\left\{\frac{D}{37279554} = -e\right.$

Operation.

37279554) ,000000026474745106) (,15)710167967 = −*e*

260956878

37905730

37279554

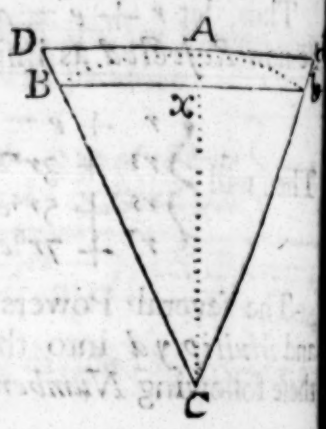
62617660

37279554

&c.

Last $r = ,000000024327$
$-e = ,000000000000000710167967$
$r - e = ,000000024326999289832033 = a$ the *Chord* or *Side* of the *Polygon* requir'd.

The next Work will be to examine how many Places of these *Figures* will hold true to the *Circle's Periphery*: In order to that let a be represented by the *Chord* $B\,b$, in the *Annexed Scheme*; and let $Bx = xb$. Then will $Bx = \frac{1}{2}a = (,7)121634996449160165$
And $\square\, BC - \square\, Bx = \square\, Cx$.
Let the *Radius* $BC = 1$ as before.
Then will the $\sqrt{\square\, BC - \square\, Bx} = Cx$
$= ,9999999999999999$, &c.



But $Cx : xB :: CA : AD$ } Per *Fig.*
Or $Cx : Bb :: CA : Dd$ }
Ergo $Dd = (,7)243269992898320354$ the Side of the *Circumscribing Polygon*, Then will $a \times 258280326$ be the *Perimeter* of the *Inscrib'd Polygon*.

And $Dd \times 258280326$ will be the *Perimeter* of the *Circumscribing Polygon*.

That is, $6,2831853071795859 =$ the *Perimeter* of the *Inscrib'd Polygon*.

And, $6,2831853071795865 =$ the *Perimeter* of the *Circumscrib'd Polygon*.

Hence 'tis evident, that the *Circle's Periphery*, whose *Diameter* is 2, may be concluded $6,2831853071795864$ true, because the *Perimeters* of the *Inscrib'd* and *Circumscrib'd Polygon* are so far very near being *Co-incident*, or the same.

'Tis possible there may be some who will think this is tedious and troublesome Work; but if those please to consider that if this *Periphery* were to be found by the aforesaid *Method* of *Bisection*, it would require these following *Extractions*.

$$\text{Viz.}\left\{\begin{array}{l} \sqrt{:2} - \sqrt{:2} + \sqrt{:2} + \sqrt{:2} + \sqrt{:2} + \sqrt{:2} + \sqrt{:2} \\ + \sqrt{:2} + \sqrt{:2} + \sqrt{:2} + \sqrt{:2} + \sqrt{:2} + \sqrt{:2} + \sqrt{:2} \\ + \sqrt{:2} + \sqrt{:2} + \sqrt{:2} + \sqrt{:2} + \sqrt{:2} + \sqrt{:2} + \sqrt{:2} \\ + \sqrt{:2} + \sqrt{:2} + \sqrt{:2} + \sqrt{:2} + \sqrt{:2} + \sqrt{:3} \text{ Multiply } d \text{ into } 402809984. \end{array}\right.$$

Here the first Root (*viz.* $\sqrt{3}$ must be *Extracted* at least one hundred and two Places of *Figures*. The second Root (*viz.*

(*viz.* $\sqrt{:2+\sqrt{3}}$) must have 99 Places of *Figures* in it. The third Root (*viz.* $\sqrt{:2+\sqrt{:2+\sqrt{3}}}$) must have 96 Places in it, &c every *Extraction* being allow'd to *Decreaſe Three Places*, that ſo the laſt *Root* (*viz.. the Chord Sought*) may conſiſt of 24 Places of *Figures*, As above.

I ſay, whoever duly conſiders the Trouble of theſe ſo often repeated *Extractions*, will, I preſume, be pleas'd with what I have done. For truly when I conſider of the great Time and Care requir'd in them, I cannot but admire at the Patience of the Laborious *Van Culen*, who proceeded that Way until he had found the *Circle's Periphery* to *Thirty Six Places* of *Figures*, to wit, 6,283185307179586476925286766559005 76.

Theſe Numbers are ſaid to be engraven upon his Tomb-Stone in St. Peter's *Church in* Leyden, *for a Memorial of ſo great a Work.*

Having thus obtain'd the *Circle's Periphery*, its *Area* may eaſily be found (*to the ſame Number of Figures*) by *Problem* 6. That is, if *Half the Periphery* of any *Circle* be *Multiply'd* into *Half* its *Diameter*, the *Product* will be that *Circle's Area*, as will appear farther on. Therefore 3,14159265358979 3 will be the *Area* of the *Circle* whoſe *Diameter* is 2.

Thus I have ſhew'd the young *Geometer* how to find the *Circle's Periphery* and *Area* to what Exactneſs he pleaſes to approach; for preciſely true they cannot be Sound, notwithſtanding the late *Pretenſions* of a certain *French-man*, who hath publiſh'd to the World, (in the *Works of the Learned*) that after twenty five Years *Study* he had found the *Quadrature* of the *Circle*: But if he had perus'd the 83 *Chapter* of Dr. *Wallis's Algebra*, he might there have ſeen his *Error*, *viz.* the *Impoſſibility* of what he pretended to; for it is as impoſſible to ſquare the *Circle* (*that is, to find its true Area*) as it is to find the *Root* of a *Surd Number*.

Note, *What I have here propos'd and done by the Triſection of an Arch, may as eaſily and much more ſpeedily be perform'd by Quinqueſection or Septiſection*, &c. *But becauſe the Scheme for Triſection is more ſimple, and may be eaſier underſtood by a Learner than thoſe of the other Sections*, (*of which ſee my* Compendium of Algebra, *Pag's 76 and 79*) *I have for that Reaſon made Choice of Triſection.*

As to the *Proportion* of one *Circle* to another, and of the *Circle* to the *Ellipſis*, &c. thoſe ſhall be fully ſhew'd when we come to the fifth Part.

Before I conclude this Part, I ſhall make ſome *Uſe* or *Application* of the above-found *Periphery*, in finding the *Quantity* of *Angles*, which is done by the Help of *Right-lines*, call'd *Sines* and *Tangents*, the Length whereof are *Calculated* to every *Degree* and *Minute* of a *Quadrant*, by much Labour. But I ſhall here ſhew how to find the *Natural Side (and conſequently the Natural Tangent)* of any propos'd *Arch* or *Angle*, by *Two Æquations*, without the Help of any *precedent Sine*, as uſual; which I did ſome Years ago communicate to the Ingenious Mr. *Joſeph Ralphſon*, and he ſo well approv'd of them as to make them the 20th and 21 *Problems* in the *Second Edition* of his *Analyſis Æquatium Univerſalis*.

And becauſe in finding the *Quantity* of *Angles*, every *Circle* is ſuppos'd to be *Divided* into 360 *Equal Parts*, called *Degrees*; every *Degree* is *Subdivided* into 60 *Parts*, call'd *Minutes*; and every *Minute* into 60 *Seconds*, &c. (See *Page* 294.)

Therefore 360) 6,2831853 &c. (0,0174532925 &c. is an *Arch* of the above-found *Periphery*, Equal to the *Arch* of one *Degree*.

And 60) 0,0174532925 &c. (0,0002908882 &c. = the *Arch* of one *Minute*.

Then if the given *Arch (or Angle)* be Leſs than 45 *Degrees*, *Reduce* it into *Minutes*, and *Multiply* thoſe *Minutes* into this conſtant *Multiplicator*, *viz.* 0,0002908882 calling the *Product p*. And for the *Sine* ſought put *a*. Then it will be

$$-aaaa + 12paaa - 195aa - 36ppaa + 240pa = 45pp.$$

Example.

Let it be requir'd to find the Sine of 19° . 13′ = 1153′.

Here 0,0002908882 × 1153 = 0,3353940946 = *p*.

And $-a^4 + 4{,}024729a^3 - 199{,}049611aa + 80{,}494583a =$
= 5,06201394.

Let $r + e = a$

Then
$$\begin{cases} rr + 2re + ee = aa \\ rrr + 3rre + 3ree = aaa \\ rrrr + 4rrre + 6rree = aaaa \end{cases}$$

Note, *In this Caſe the firſt* r *may always be taken Equal to the Firſt Figure in the Product* = p. Viz. *here* r = 0 3 *which being Involv'd as its Powers direct, and thoſe Powers Mltiply'd into the Reſpective Co-efficients of the Æquation, it will be*

$$\left\{\begin{array}{llll} + 24{,}1483 & + 80{,}49e & & \\ - 17{,}9144 & - 119{,}43e & - 199{,}05ee & \\ + \ 0{,}1086 & + \ 1{,}08e & + \ 3{,}62ee & \\ - \ 0{,}0081 & - \ 0{,}11e & - \ 0{,}54ee & \end{array}\right\} = 5{,}06201394$$

Viz. $6{,}3344 - 37{,}97e - 195{,}97ee = 5{,}06201$

Hence

Hence $37,97ee + 195,97ee = 1,27239$
And $0,193e + ee = 0,006492 = D$

$$\text{Theorem} \left\{ \frac{D}{,193 + e} = e \right.$$

Operation. 0,193) 0,006492 (0,029 = *e*

$+e =$	,029	42
1. *Divisor*	,21	2292
2. *Divisor*	,222	1998

First $r = 0,3$
$+e = 0,029$
$r + e = 0,329 = r$ for a *Second Operation.*

Which being *Involv'd* and *Multiply'd*, *&c.* as before, will *produce* these *Numbers.*

$+ 26,48271781 + 80,49458e$
$- 21,54532894 - 130,97464e - 199,0496ee$
$+ 0,14332578 + 1,30692e + 3,9724ee$
$- 0,01171611 - 0,14244e - 0,6494ee$

Viz. $5,06899854 - 49,31558e - 195,7266ee - 5,06201394$

Hence $49,31558e + 195,7266ee = ,0069846$; which being *Divided* by 195,7266 the *Co-efficient* of *ee* will become

$$,25196e + ee = ,0000356854 = D$$

$$\text{Then} \left\{ \frac{D}{,25196 + e} = e \right.$$

Operation. 0,25196) ,0000356854 (0,0001415 = *e*

$+e =$	0,00014	2520
1. *Divisor*	0,2520	104854
2. *Divisor*	0,25210	100840
		40140
		25210
Last $r =$	0,329	&c.
$+e =$	0,0001415	

$r + e = a = 0,3291415$ being the *Natural Sine of* 90° . 13'
As was requir'd.

Thus you may find the *Right Sine* of any *Arch* or *Angle less* than 45 *Degrees.*

But

But if the given *Arch* be *Greater* than 45 *Degrees*, you muſt take its *Complement* to 90°. *viz. ſubſtract* it from 90 *Degrees*, and *Reduce* the *Remainder* into *Minutes*, As before.

Then *Multiply* the *Square* of thoſe *Minutes* into this *conſtant Multiplicator*, 0,000000084616 calling their *Product* p, and putting $a =$ the *Sine ſought*, As before. Then will

$$a^4 + 28a^3 + 195aa + 36paa + 108pa - 28a = 196 - 81p$$

Example.

Suppoſe it were requir'd to find the *Sine* of 75°. 32'. (Or *which is the ſame thing*) to find the *Co-ſine* of 14° . 28'. = 868' whoſe *Square* 753424 × 0,000000084616 = 0,06375172518 = p. Hence the *Æquation* in *Numbers* will be

$$aaaa + 28aaa + 197{,}295062aa - 21{,}114814a = \\ = 190{,}8361102588$$

Let $r - e = a$ And $r = 1$

$$\text{Then} \begin{cases} rr - 2re + ee = aa \\ rrr - 3rre + 3ree = aaa \\ rrrr - 4rrre + 6rree = aaaa \end{cases}$$

Note, *I here take* $r = 1$ *becauſe the Arch is ſo near to* 90°. *and therefore I make it* $r - e = a$.

$$\text{Then} \left\{ \begin{array}{rrrr} - & 21{,}1148 & + 21{,}11e & \\ + & 197{,}2956 & - 394{,}59e + & 197{,}29ee \\ + & 28{,}0000 & - 84{,}00e + & 84{,}00ee \\ + & 1{,}0000 & - 4{,}00e + & 6{,}00ee \end{array} \right\} = 190{,}8361$$

Viz. $205{,}1808 - 461{,}48e + 287{,}29ee = 190{,}8361$

Hence $461{,}48e - 287{,}29ee = 14{,}3447$

And $1{,}606e - ee = {,}049930 = D$

$$\text{Theorem} \left\{ \frac{D}{1{,}606 - e} = e \right.$$

Operation. 1,606) ,049930 (0,031 = e

$-e = 0{,}031$ 471

1. *Diviſor*	1,57	2830	First $r = 1{,}000$
2. *Diviſor*	1,575	1575	$- e = 0{,}031$
		&c.	$r - e = 0{,}969 = r$

For a *Second Operation*; which being *Involv'd* as before, will *produce* theſe following *Numbers*.

— 29

$$
\begin{array}{l}
-\ 20{,}460254766 + 21{,}11481e \\
+\ 185{,}252368710 - 382{,}35783e + 197{,}2951ee \\
+\ 25{,}475889852 - 78{,}87272e + 81{,}5960ee \\
+\ 0{,}881647759 - 3{,}63941e + 5{,}6337ee \\
\hline
\end{array}
$$

Viz. $191{,}149651515 - 443{,}75515e + 284{,}5248ee =$
$= 190{,}836110259$

Hence it will be $443{,}75515e - 284{,}5248ee = 0{,}313541256$

And $1{,}55963e - ee = {,}0011019821 = D$

$$\text{Then}\left\{\frac{D}{1{,}55963 - e} = e\right.$$

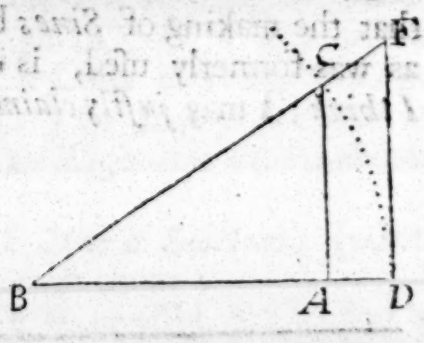
peration.	1,55963)	0,0011019821
$-e =$	0,00070	109123
Divisor	1,5589	1075210
Divisor	1,55893	935358
		1398520
Last $r = 0{,}969$		1247144 &c.
$-e = 0{,}0007068$		

$-e = a = 0{,}9682932$ the *Sine* of 75°. 32′. As was requir'd.

Having found the *Sine* and *Co-sine* of any *Arch*, *the Tangent* usually found by this *Proportion*;

iz. { *As the Co-sine of any Arch : is to the Sine* of the *Arch* :: *so is the Radius : to the Tangent of the same Arch.*

For supposing $BC = BD$ *Radius*, AC the *Sine* of the *Arch* D. Then BA is the *Co-sine*, and FD the *Tangent* of the same *Arch*. ut $BA : CA :: BD : FD$, &c.



Now by this *Proportion* there is *equir'd* to be given both the *Sine* nd *Co-sine* of the *Arch*, to nd the *Tangent*.

'Tis true, if the *Radius*, and ither the *Sine* or the *Co-sine* be given, the other may be found Thus, $\sqrt{\square BC - \square CA} = BA$. Or $\sqrt{\square BC - \square CA} = CA$.

But if either the *Sine* or *Co-sine* be given, the *Tangent* may (*I presume*) be more *Easily* found by the following *Theorems*.

Let

Let $BC = 1$. $CA = S$. $BA = x$, and $FD = T$. Then if S be given, T may be found by this

$$\text{Theorem} \left\{ \sqrt{\frac{SS}{1 - SS}} = T \right.$$

Or if x be given, T may be found by this

$$\text{Theorem} \left\{ \sqrt{\frac{xx}{1 - xx}} = T \right.$$

Let the *Sine* of 19° . 13′. *(before found)* be given, viz. 0,3291415 = S. To find T the *Tangent* of the same *Arch*.
First 0,3291415 × ,3291415 = 0,108334127 = SS.
Again 1 − 0,108334127 = 0,891665873 = $1 - SS$.
Then 0,891665873) 0,108334127 (0,1214963253
And $\sqrt{}$0,1214963253 = 0,3485632 = T, the *Tangent* of 19° . 13′. As was requir'd.

And so you may *proceed* to find T = the *Tangent*, when x = the *Co-sine* is given.

Perhaps it may here be *expected*, that I should have shewn and *Demonstrated (or at least have inserted)* the *Proportion* from whence the foregoing *Æquations* for making *Sines* were *produc'd*; but I have *Omitted* that, as also their *Use* in *computing* the *Sides* and *Angles* of *plain Triangles* by the *Pen* only, *(viz. without the Help of Tables)* for the *Subject* of any *Discourse* hereafter, if *Health* and *Time* permit.

In the mean *Time*, what is here done may suffice to shew that the making of *Sines* by such a *Laborious* and *Operose* Way as was formerly used, is in a great *Measure* overcome; which *I think*; I may *justly claim* as my *own*.

AN INTRODUCTION TO THE Mathematicks.

PART IV.

CHAP. I.

Definitions of a Cone, *and its* Sections.

THERE are several Definitions given of a *Cone:* The Learned Dr. *Barrow*, upon *Euclid*, hath it thus:

"A Cone *(saith he)* is a Figure made when one Side of a Rectangled Triangle, (*viz.* one of those Sides that contain the Right-angle) remaining fix'd, the Triangle is turn'd round about, 'till it return to the place from whence it first mov'd: And if the fix'd Right-Line be equal to the other which containeth the Right-angle, then the Cone is a Rectangled Cone; but if it be less, 'tis an Obtuse-angled Cone; if greater, an Acute-angled Cone. The *Axis* of a Cone is that fix'd Line above which the Triangle is mov'd: The *Base* of a Cone is the Circle, which is describ'd by the Right-line mov'd about. (*Defin.* 18, 19, 20. *Euclid.* 11.)

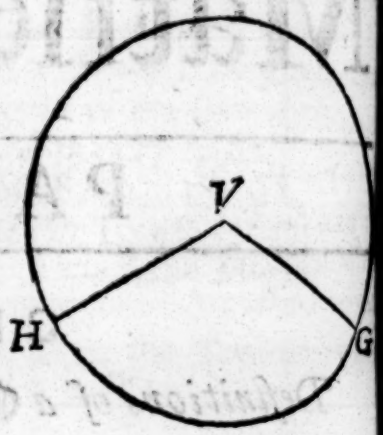
Sir *Jonas Moor*, in his Treatise of *Conical Sections*, (taken out of the Works of *Mydorgius*) defines it thus:

"If a Line of such a length as shall be needful, shall upon a Point fix'd above the Plain of a Circle, so move about the Circle, until it return to the Point from whence the Motion began, the Superficies that is made by such a Line is call'd a *Conical Superficies*; and the solid Figure contain'd within that Superficies and the Circle is call'd a *Cone*. The Point remaining still is the Vertex of the Cone, &c.

Altho' both these *Definitions* are equally true, and, with a little Consideration, may be pretty easily understood, yet I shall here propose one very different from either of them, and, as I presume, more plain and intelligible, especially to a *Learner*.

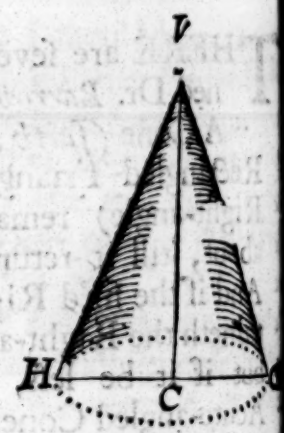
If a Circle describ'd upon stiff Paper (*or any other pliable Matter*) of what bigness you please, be cut into *Two*, *Three*, or more *Sectors*, either equal or unequal, and one of those *Sectors* be so roll'd up, as that the *Radius's* may exactly meet each other, it will form a *Conical Superficies*.

That is, if the *Sector HVG* be cut out of the Circle, and so roll'd up as that of the *Radius's VH* and *VG* may just meet each other in all their Parts, it will form a *Cone*, and the Centre *V* will become a *Solid Point*, call'd the VERTEX of the *Cone*; the *Radius VH* being every where Equal, will be the Side of the Cone, and the Arch *HG* will become a Circle, whose *Area* is call'd the *Cone's Base*.



A *Right-line* being suppos'd to pass from the *Vertex*, or Point *V*, to the *centre* of the Cone's *Base*, as at *C*, that Line (viz. *VC*) will be the AXIS, or *perpendicular Height* of the Cone.

If a *Solid* be exactly made in such a Form, it will be a compleat or perfect Cone; which I shall all-along call a Right Cone, because its *Axis VC* stand at *Right Angles* with the Plain of its *Base HG*, and its Sides are every where equal.



Any Cone whose *Axis* is not at *Right-angles* with the Plain of its *Base*, may be properly call'd an *Imperfect Cone*, because its Sides are not every where equal (*as in the annexed Figure*). Now, such an imperfect Cone is usually call'd a *Scalene*, or *Oblique Cone*.

Any Solid Cone may be cut by Plains (*which I shall all-alo hereafter call Right-lines*) into *Five Sections*.

Sect. 1.

If a Right Cone be cut directly thro' its *Axis*, the *Plain* or *Superficies* of that *Section* will be a plain *Isosceles Triangle*, as *HVG Fig.* 2, viz. the Sides (*HV* and *VG*) of the Cone will be the Sides of the Triangle, the Diameter (*HG*) of the Cone's *Base* will be the *Base* of the Triangle, and (*VC*) its Axis will be the *Perpendicular Height* of the Triangle.

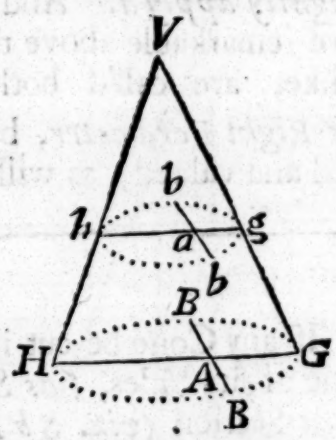
Sect. 2.

If a Right Cone be cut (*any where*) off by a Right-line parallel to its *Base*, as *h g* (*it will be easie to conceive, that*) the Plain of that *Section* will be a Circle, because the Cone's *Base* is such: Wherein one thing ought to be clearly understood, which may be laid down as a *Lemma*, to demonstrate the Properties of the following *Sections*.

Lemma. { *If any Two Right-lines, inscrib'd within a Circle, do cut or cross each other* (as *h g* doth *b b* in the annexed Figure) *the Rectangle made of the Segments of one of the Lines will be Equal to the Rectangle made of the Segments of the other Line.* (See *Theorem* 15. *Page* 315.

That is, $ha \times ga = ba \times ab$
And $HA \times GA = BA \times AB$ } &c.

Consequently if $ba = ab$
And if $BA = AB$
Then it will be $ha \times ga = \square ba$
And in the *Cone*'s Base $HA \times GA = \square BA$.



Sect. 3.

If a *Right Cone* be (*any where*) cut *Off* by a *Right Line* that cuts both its *Sides*, but not *parallel* to its *Base*, (*as T S in the following Figure*) the Plain of that Section will be an *Ellipsis*, (*vulgarly call'd an* Oval) *viz.* an oblong or imperfect Circle, which hath several *Diameters*, and *Two particular Centres*. That is,

1. Any *Right Line* that divides an *Ellipsis* into *Two* Equal Parts is call'd a *Diameter*; amongst which, the *longest* and the *shortest* are particularly distinguish'd from the rest, as being of most general use; the other are only applicable to particular Cases.

2. The *longeſt Diameter* (as *TS*) is call'd the *Tranſverſe Diameter*, or *Tranſverſe Axis*, being that *Right-line* which is drawn thro' the middle of the *Ellipſis*, and doth ſhew or limit its length.

3. The *ſhorteſt Diameter*, call'd the *Conjugate Diameter*, is a *Right-line* that doth interſect or croſs the *Tranſverſe Diameter* at *Right-angles*, in the middle or common Centre of the *Ellipſis*, (as *Nn*) and doth limit the *Ellipſis Breadth*.

4. The *Two Points*, which I call particular *Centres* of an *Ellipſis* (*for a Reaſon which ſhall be ſhew'd farther on*) are two Points in the *Tranſverſe Diameter*, at an equal diſtance each way from the *Conjugate Diameter*, and are uſually call'd **Nodes**, **Focus's**, or *burning Points*.

5. All *Right-lines* within the *Ellipſis* that are parallel to one another, and can be divided into *Two* equal Parts, are call'd **Ordinates**, with reſpect to that *Diameter* which divides them: And if they are parallel to the *Conjugate*, viz. at *Right-angles* with the *Tranſverſe Diameter*, then they are call'd *Ordinates rightly apply'd*. And thoſe *Two* that paſs through the *Focus's* are remarkable above the reſt, which, being equal and ſituated alike, are call'd both by one Name, *viz.* **Latus Rectum**, or *Right Parameter*, by which all the other *Ordinates* are regulated and valued; as will appear farther on.

Sect. 4.

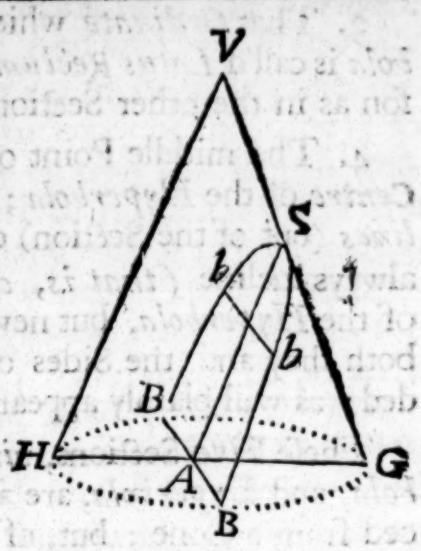
If any Cone be cut into *Two parts* by a *Right-line* parallel to one of its Sides, (*as S A in the following Scheme*) the Plain of that Section (*viz. S b B A B b S*) is call'd a **Parabola**.

1. A *Right-line* being drawn thro' the middle of any *Parabola* (as *S A*) is call'd its *Axis*, or intercepted Diameter.

2. All *Right-lines* that interſect or cut the *Axis* at *Right Angles* (*as B B and b b are ſuppos'd to cut or croſs S A*) are call'd *Ordinates rightly apply'd*, (*as in the Ellipſis*) and the greateſt *Ordinate*, as *B B*, which limits the *length* of the *Parabola*'s *Axis* (*S A*) is uſually call'd the Baſe of the *Parabola*.

3. That

3. That *Ordinate* which paſſes thro' the *Focus*, or burning Point of the *Parabola*, is call'd the *Latus Rectum*, or *Right Parameter*, *(as in the Ellipſis)* becauſe by it all the other *Ordinates* are *proportion'd*, and may be found.

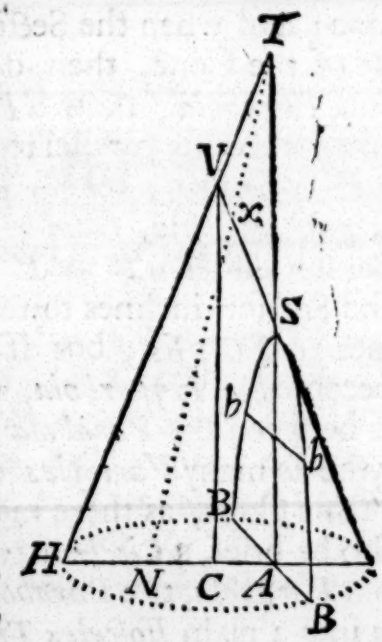


4. The *Node*, *Focus*, or *burning Point* of the *Parabola*, is a Point in its Axis *(but not a Centre, as in the Ellipſis)* diſtant from the *Vertex*, or Top of the Section, (viz. *from S)* juſt ¼ part of the *Latus Rectum*; as ſhall be ſhewn further on.

5. All *Right-lines* drawn within a *Parabola* parallel to its *Axis* are call'd *Diameters*; and every *Right-line* that any of thoſe *Diameters* doth biſect or cut into Two equal Parts, is ſaid to be an *Ordinate* to that *Diameter* which biſects it.

Sect. 5.

If a Cone be any where cut by a *Right-line*, either parallel to its *Axis*, *(as S A, or otherwiſe as x N)* ſo as the cutting Line being continued thro' one Side of the Cone (as at *S* or *x)* will meet with the other Side of the Cone, if it be continued or produc'd beyond the Vertex *V*, as at *T*; then the Plain of that Section (viz. *the Figure S b B B b S)* is call'd an **Hyperbola.**



1. A *Right-line* being drawn thro' the middle of any *Hyperbola*, viz. within the Section, *(as S A, or x N)* is call'd the *Axis* or intercepted *Diameter*, (*as in the Parabola*) and that part of it which is continued or produced out of the Section, until it meet with the other Side of the Cone continued, viz. *T s* or *T x*, &c. is call'd the *Tranſverſe Diameter*, or *Tranſverſe Axis* of the *Hyperbola.*

2. All *Right-lines* that are drawn within an *Hyperbola*, at *Right-angles* to its *Axis*, are call'd *Ordinates rightly apply'd*; as in the *Ellipſis* and *Parabola.*

3. That

3. That *Ordinate* which passes thro' the *Focus* of the *Hyperbola* is call'd *Latus Rectum*, or *Right Parameter*, for the same reason as in the other Sections.

4. The middle Point of the *Transverse Diameter* is call'd the *Centre* of the *Hyperbola*; from whence may be drawn *two Right-lines* (out of the Section) call'd **Asymptotes**, because they will always incline (*that is, come nearer and nearer*) to both Sides of the *Hyperbola*, but never meet with (or touch) them, altho' both they and the Sides of the *Hyperbola* were infinitely extended; as will plainly appear in its proper place.

These Five Sections, *viz.* the *Triangle*, *Circle*, *Ellipsis*, *Parabola*, and *Hyperbola*, are all the Plains that can possibly be produced from a Cone; but, of them, the *Three Last* are only call'd *Conick Sections*, both by the ancient and modern *Geometers*.

Scholium.

Besides the 'foregoing *Definitions*, it may not be amiss to add, by way of Observation, how one Section may (*or rather doth*) change or degenerate into another.

An *Ellipsis* being that Plain of any Section of the Cone which is between the *Circle* and *Parabola*, 'twill be easie to conceive that there may be great variety of *Ellipses* produc'd from the same Cone; and when the Section comes to be exactly parallel to one Side of the Cone, then doth the *Ellipsis* change or degenerate into a *Parabola*. Now a *Parabola* being that Section whose Plain is always exactly parallel to the Side of the Cone, cannot vary, as the *Ellipsis* may; for so soon as ever it begins to move out of that *position*, (viz. *from being parallel to the Cone's Side*) it degenerates either into an *Ellipsis*, or into an *Hyperbola*: That is, if the Section inclines towards the Plain of the Cone's Base, it becomes an *Ellipsis*; but if it incline towards the Cone's Verter, it becomes an *Hyperbola*, which is the Plain of any Section that falls between the *Parabola* and the *Triangle*. And therefore there may be as many *Varieties* of *Hyperbola's* produc'd from one and the same Cone, as there may be *Ellipses*.

To be brief, a *Circle* may change into an *Ellipsis*, the *Ellipsis* into a *Parabola*, the *Parabola* into an *Hyperbola*, and the *Hyperbola* into a plain *Isosceles Triangle*: And the Centre of the Circle, which is its *Focus* or burning Point, doth, as it were, part or divide it self into two *Focus's* so soon as ever the Circle begins to degenerate into an *Ellipsis*; but when the *Ellipsis* changes into a *Parabola*, one end of it flies open, and one of its *Focus's* va-

nishes,

niſhes, and the remaining *Focus* goes along with the *Parabola* when it degenerates into an *Hyperbola:* And when the *Hyperbola* degenerates into a plain *Iſoſceles Triangle*, this *Focus* becomes the *vertical* Point of the *Triangle* (*viz.* the *Vertex* of the Cone); ſo that the Centre of the Cone's Baſe may be truly ſaid to paſs gradually thro' all the Sections, until it arrives at the *Vertex* of the Cone, ſtill carrying its *Latus Rectum* along with it: For the *Diameter* of a Circle being that *Right-line* which paſſes thro' its *Centre* or *Focus*, and by which all other *Right-lines* drawn within the Circle are regulated and valued, may (*I preſume*) be properly call'd the Circle's *Latus Rectum:* And altho' it loſes the Name of *Diameter* when the Circle degenerates into an *Ellipſis*, yet it retains the Name of *Latus Rectum*, with its firſt Properties, in all the Sections, gradually ſhortening as the *Focus* carries it along from one Section to another, until at laſt *it* and the *Focus* become co-incident, and terminate in the *Vertex* of the Cone.

I have been more particular and fuller in theſe *Definitions* than is uſual in Books of this Subject, which I hope is no Fault, but will prove of Uſe, eſpecially to a *Learner:* And altho' they may perhaps ſeem a little ſtrange, and at firſt hard to be underſtood, yet when they are well conſider'd, and compar'd with a Cone cut into ſuch Sections as have been defined, they will not only be found true, but will alſo help to form a true and clear *Idea* of each Section.

CHAP. II.

Concerning the Chief Properties of an Ellipsis.

Note, *If the tranſverſe Diameter of an* Ellipſis, *as TS in the following* Figure, *be interſected or divided into any two parts by an* Ordinate rightly apply'd, *as at the Points A, C, a,* &c. *Then are thoſe parts TA, TC, Ta, and SA, SC, Sa,* &c. *uſually call'ed* Abſciſſa's, (*which ſignifies Lines or Parts cut off*) *and by the Rectangle of any two* Abſciſſa's *is meant the Rectangle of ſuch two parts as, being added together, will be equal to the Tranverſe Diameter.*

As $TA + SA = TS$. And $TC + SC = TS$.
Or $TA + SA = TS$, &c.

Sect.

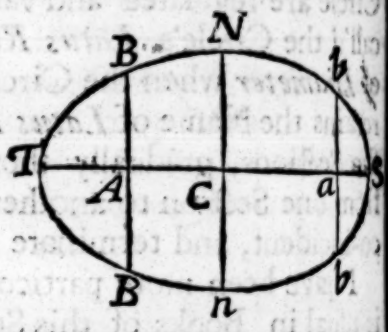
Section 1.

EVERY *Ellipsis* is proportion'd, and all such Lines as relate to it are regulated, by the help of one general *Theorem*.

Theorem { *As the Rectangle of any two* Abscissa's : *is to the Square of any* Ordinate *which divides them* :: *So is the Rectangle of any other two* Abscissa's : *To the Square of half that* Ordinate *which divides them.*

That is,

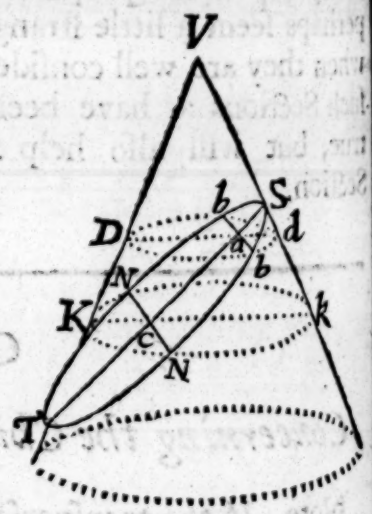
$TA \times SA : \square BA :: Ta \times Sa : \square ba$
$TA \times SA : \square BA :: TC \times SC : \square NC$
$TC \times SC : \square NC :: Ta \times Sa : \square ba$
&c.



Demonstration.

Let the annex'd Figure represent a Right Cone, cut thro' both Sides by the Right-line TS; then will the Plain of that Section be an *Ellipsis*, (by *Sect.* 3. *chap.* 1.) TS will be the *Transverse Diameter* NCN and $b\,a\,b$ will be *Ordinates rightly apply'd*; as before.

Again, if the Lines Dd and Kk be parallel to the Cone's Base, they will be Diameters of Circles (by *Sect.* 2, *chap.* 1.) Then will $\triangle TCK$ and TaD be alike. Also $\triangle Sad$ and $\triangle SCk$ will be alike.



Ergo	1	$Sa : ad :: Sc : Ck$	} per *Theorem* 13.
And	2	$TC : CK :: Ta : aD$	
1	∴	3 $Sa \times Ck = ad \times SC$	
2	∴	4 $Ta \times CK = TC \times aD$	
2 × 3		5 $Sa \times CK \times Ta \times CK = ad \times SC \times TC \times aD$.	Per *Axiom* 3.
But		6 $CK \times Ck = \square NC$	} per *Lemma Sect.* 2.
And		7 $aD \times ad = \square ba$	
Then		for $CK \times Ck$, and $aD \times ad$, take $\square NC$ and $\square ba$	
5, 6, 7		8 $Sa \times Ta \times \square NC = TC \times SC \times \square ba$	Per *Axiom* 5.
Hence		9 $Sa \times Ta : \square ba :: TC \times SC : \square NC$.	See page 194.

Q. E. D.

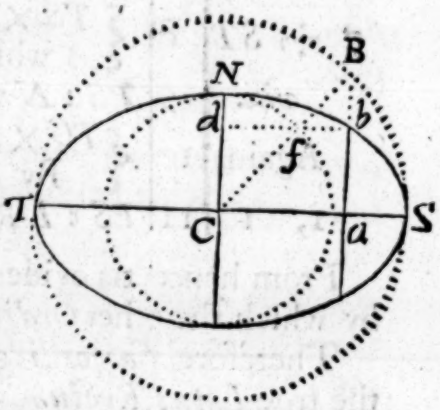
Or

Or, the Truth of these Proportions may be otherwise prov'd by a Circle, without the help of the Cone; thus:

Let any *Ellipsis* be *circumscrib'd* and *inscrib'd* with Circles, as in the following Figure; then from any Point in the circumscrib'd Circle's Periphery, as at *B*, draw the Right-line *B a*, parallel to the *semi-conjugate* Diameter *NC*, then will *b a* be a *Semi-ordinate rightly apply'd* to the Transverse Diameter *T S*; as before.

Again, From the Point *b* *(in the Ellipsis Periphery)* draw the Right-line *b d* parallel to the Transverse *T S*; and draw the *Radius B C*.

Then will $\triangle B C a$ and $\triangle C f d$ be alike.



Therefore	1	$BC : Ba :: Cf : dC$ per *Theorem* 13.
But	2	$TC - BC, NC = Cf$ and $ba = dC$
Conseq.	3	$TC : Ba :: NC : ba$
Or	4	$TC : NC :: Ba : ba$
4 in □'s	5	$\square TC : \square NC :: \square Ba : \square ba$
But	6	$Ta \times Sa = \square Ba$ per Lem. *sect.* 2 *chap.* 1
Therefore	7	$Ta \times Sa : \square ba :: TC \times SC = \square TC : \square NC$, as before.

And so for any other *Abscissa's*, and their *Semi-ordinate*.

These Proportions being found to be the true and common Properties of every *Ellipsis*, all that is farther requir'd in *(or about)* that *Section* may be easily deduced *from* them.

Sect. 2. *To find the* **Latus Rectum**, *or* **Right Parameter** *of any Ellipsis.*

There are several Ways of finding the *Latus Rectum*, but I think none so easie, and shews it so plainly to be the Third principal Line in the *Ellipsis*, as the following

Theorem. *As the* Transverse Diameter : *is in Proportion to the* Conjugate : : *so is the* Conjugate : *to the* Latus Rectum.

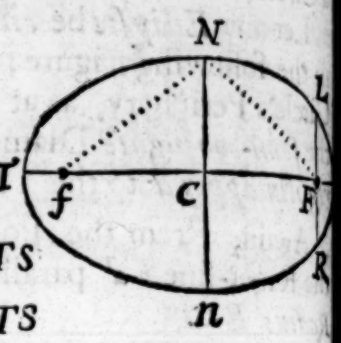
Viz. *(in the following Fig.)* $TS : Nn :: Nn : LR$ the *Latus Rectum*.

Demonstration.

From the last *Proportions* take either of the Antecedents, and its Consequent, *viz.* either $TC \times SC : \square NC$, or $Ta \times Sa : \square ba$,

and make TS the Third Term, to which find a Fourth Proportional, and it will be $= LR$:

Thus	1	$TC \times SC : \square NC :: TS : LR$
But	2	$TC = SC$ And $NC = Cn$
Therefore	3	$TC \times SC = \frac{1}{4} TS$
And	4	$\square NC = \frac{1}{4} \square Nn$
1, 3, 4	5	$\frac{1}{4} \square TS : \frac{1}{4} \square Nn :: TS : LR$
5 ∴	6	$\frac{1}{4} \square TS \times LR = \frac{1}{4} \square Nn \times TS$
6 × 4	7	$\square TS \times LR = \square Nn \times TS$
7 ÷ ST	8	$TS \times LR = \square Nn$ which gives the following *Analogy*.
viz.	9	$TS : Nn :: Nn : LR$
Again,	10	$TC \times SC : \square NC :: Ta \times Sa : \square ba$ By *common Properties*.
1, 10	11	$TS : LR :: Ta \times Sa : \square ba$.



From hence 'tis evident that LR, thus found, is that *Ordinate* by which the other *Ordinates* may be regulated and found.

Therefore (*according to its Definition* Sect. 3, Chap. 1.) it is the true *Latus Rectum*. Q. E. D.

Consectary.

Hence it follows, that if the Transverse and Conjugate Diameters of any *Ellipsis* are given (*either in Lines or Numbers*) the *Latus Rectum* may be easily found; and then any *Ordinate*, whose Distance from the Conjugate is given, may be found; As above.

Sect. 3. *To find the* Focus *of any* Ellipsis.

The *Focus* is the Distance of the *Latus Rectum* from the Conjugate or Middle of the *Ellipsis*, (*vide* Definition 4, page 358.) and that Distance is always a *Mean Proportional* between the Half-Sum and Half-Difference of the Transverse and Conjugate Diameters, which gives this *Theorem*.

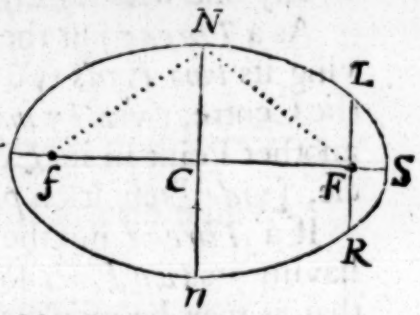
Theorem. *From the* Square *of half the* Transverse *subtract the* Square *of half the* Conjugate, *the square Root of their difference will be the Distance of each* Focus *from the middle or common centre of the* Ellipsis.

That is, supposing the Points f and F to be the two *Focus's*, viz. $fC = CF$, and $TC = \frac{1}{2} TS$. $NC = \frac{1}{2} Nn$. Then, $TC + NC : fC :: FC : TC$. Ergo $\square FC = \square TC - \square NC$. Consequently, $FC = \sqrt{\square TC - \square NC}$.

Demon-

Demonstration.

First, 1 $TS \times LR = \square Nn$. By 8 Steps of the last Process.
And 2 $TS : LR :: TF \times SF : \square LF$. Per *common Properties*
That is, 3 $TS : LR :: TC + CF \times TC - CF : \frac{1}{4}\square LR = \square LF$
3 ∴ + 4 $\left\{ \begin{array}{l} \frac{1}{4}\square LR \times TS = \\ \square TC - \square CF \times LR \end{array} \right.$
÷ LR 5 $\frac{1}{4} LR \times TS = \square TC - \square CF$
÷ 4 6 $\frac{1}{4} TS \times LR = \frac{1}{4}\square Nn = NC$
6, 6 7 $\square NC = \square TC - CF$
7, + 8 $\square CF = \square TC - NC$
3 ww 2 9 $CF = \sqrt{\square TC - \square NC}$



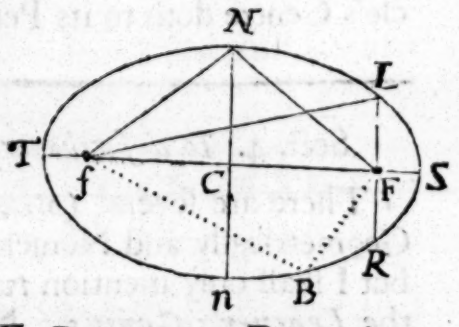
Now from hence is deduced that Notable Proposition, upon which is grounded the usual Method of describing an *Ellipsis*, and drawing of *Tangents*, &c.

Proposition *If from the two* Focus's *of any* Ellipsis *there be drawn two* Right-lines, *so as to meet each other in any point of the* Ellipsis's Periphery, *the* Sum *of those* Lines *will be Equal to the* Transverse.

Viz. $fN \times NF = TS$. $fL + LF = TS$. Or $fB \times BF = TS$, &c.

Demonstration.

First, 1 $\left\{ \begin{array}{l} \square CF \times \square NC = \square TC \\ \textit{by 8th of the last.} \end{array} \right.$
But, 2 $\left\{ \begin{array}{l} \square CF + \square NC = \square NF \\ \textit{by} \text{ Theorem 11.} \end{array} \right.$
1, 2 3 $\left\{ \begin{array}{l} \square NF = \square TC \\ \textit{by} \text{ Axiom 5.} \end{array} \right.$
ww 2 4 $NF = TC$
Hence $2NF = 2TC = TS$
Again, 5 $TS : LR :: TF \times FS : \square LF$. By common Properties.
Conseq. 6 $\frac{1}{2} TS : \frac{1}{2} LR :: TF \times FS : \square LF$
But, $\frac{1}{2} TS = TC$. And $\frac{1}{2} LR = LF$
Ergo 7 $TC : LF :: TC + CF \times TC - CF : \square LF$
7 ∴ 8 $TC \times LF = \square TC - \square CF$
But, 9 $\square fF + \square LF = \square fL$ By Theorem 11.
That is, 10 $4\square CF + \square LF = \square fL$ for $2CF := fF$
× 4 11 $4TC \times LF = 4\square TC - 4\square CF$
10 × 11 12 $4\square TC + \square LF = \overline{4TC \times LF} : + \square fL$
12 = 13 $+\square TC : - \overline{4TC \times LF} : + \square LF = \square fL$



13 *ww* 2	14	$2TC - LF = fL$
14 + *LF*	15	$2TC = fL + LF$. But $2TC = TS$
Ergo		$fL \times Lf = TS$. Q. E. D.

And this *Propoſition* muſt needs hold true to every Point in the *Ellipſis Periphery*, *viz.* at *B*, *&c.* as will evidently appear to any one who rightly conſiders, That

As a *Thread* juſt the length of the Diameter of any Circle having its *two Ends* ty'd together, and then mov'd about a Point in the Centre, (viz. *by making it a double Radius*) will, by drawing another Point in its *Extremity*, deſcribe the *Periphery* of a Circle, [*vide* Definition *page* 280] even ſo,

If a *Thread* juſt the length of the *Tranſverſe* Diameter (*TS*) having its *two Ends* ſo fix'd upon the *two Focus's* (*f* and *F*) as that it may be mov'd about them, by drawing a Point in its *Extremity* (viz. *at its full Stretch*) it will deſcribe the true Periphery of an *Ellipſis*.

Now, altho' this eaſie way of deſcribing, or, as uſually phras'd, drawing an *Ellipſis*, be mechanical, and known even to moſt *Joyners*, *Carpenters*, &c. yet it gives as compleat and clear an *Idea* of that Figure as any other way whatſoever; and by deſcribing it thus about its two *Focus's*, as a Circle is about its Centre, doth plainly ſhew thoſe two Points are not improperly call'd *particular Centres* in *Definition* 4, *Sect.* 2, *chap.* 1. for each of them bears much the ſame reſpect to the *Ellipſis* Periphery, as the Circle's Centre doth to its Periphery.

Sect. 4. *To deſcribe or delineate an* Ellipſis *ſeveral Ways.*

There are ſeveral (*other*) ways of deſcribing an *Ellipſis*, both Geometrically and Numerically, according to peculiar Occaſions; but I ſhall only mention *two* or *three* of them, leaving the reſt to the *Learner's Genius*. Now, in order to that Work, it will be convenient to conſider what *Lines* are requiſite to limit or bound its Form, which I take to be chiefly theſe following.

1. If the Tranſverſe and Conjugate are given, the *Ellipſis* is perfectly limited; (vide *Conſectary page* 363.) for if *TS* and *Nn* be ſet at *Right-angles* in their middle at *C*, and *TC* or *CS* be ſet *off* from *N*, or *n*, both Ways upon the Tranſverſe to *f* and *F* (viz. *make* $fN = TC = NF$) then will thoſe Points *f* and *F* be the *two Focus's*, (by *4th Step of the laſt Proceſs*) and then the *Ellipſis* may be deſcrib'd as above.

2. I

2. If the *Transverse Diameter* and *Latus Rectum* are given, the *Ellipsis* is truly limited, because by them the Conjugate may be found, by Sect. 2.

3. Or if only the *Transverse*, and the Proportion it hath either to the *Conjugate* or *Latus Rectum*, be given, the *Ellipsis* is thereby limited. As for instance; suppose the given *Ratio* between the Transverse and Conjugate to be, As a : to d:

Viz. $a : d :: TS : Nn$, then $\frac{TS \times d}{a} = Nn$, &c.

4. If either the Transverse or Conjugate, and the Distance of the *Focus* from the *Conjugate* be given, the *Ellipsis* is limited, because by them the Conjugate or Transverse may be found.

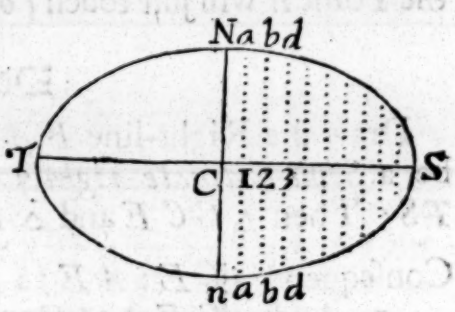
These being premis'd, and the *precedent Work* a little consider'd, it must be easie to describe or delineate any *Ellipsis* in *Plano*, either Geometrically or Numerically.

1. *To describe an* Ellipsis *Numerically*, by *Points*.

Suppose the *Transverse Diameter* $TS = 20$, and the *Conjugate* $Nn = 12$, (*either Inches, or any other Equal Parts*) and let them cross each other at *Right-angles* in their Middles, as in the Point C; then will $TC = CS = 10$.

And $NC = Cn = 6$

And it will $20 : 12 :: 12 : 7,2$ = the *Latus Rectum*.



Again $20 : 7,2$ Or rather take their *Ratio*

Thus
$1 : 0,36 :: 10 + 1 \times 10 - 1 : \square a. \parallel 1.$
$1 : 0,36 :: 10 + 2 \times 10 - 2 : \square b. \parallel 2.$
$1 : 0,36 :: 10 + 3 \times 10 - 2 : \square d. \parallel 3.$ &c.

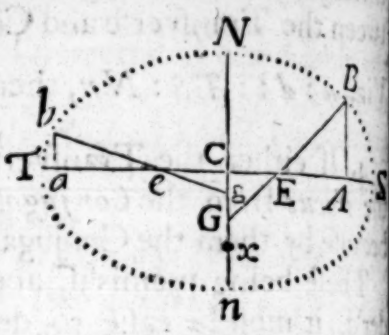
Viz.
$100 - 1 \times 0,36 = \square a, 1.$ Hence $\sqrt{99\ 0,36}$ 5,97 &c. $= a.1$
$100 - 4 \times 0,36 = \square b. 2.$ $\sqrt{96 \times 0,36} = 5,88$ &c. $= b.2$
$100 - 9 \times 0,36 = \square d. 3.$ $\sqrt{91 \times 0,36} = 5,72$ &c. $= d.3$

If so many *Semi-ordinates* as may be thought convenient (*the more the better*) be found in this manner, and every one of them be set off at *Right-angles* from its respective Point in the *Transverse Diameter* each way, *viz.* from 1 to a, from 2 to b, from 3 to d, &c. Then if a Curve Line be carefully drawn with an even Hand thro' those Extream Points a, b, d, &c. it will be the *Ellipsis Periphery* requir'd.

2. *To*

2. *To describe an* Ellipsis *Geometrically by Points.*

Having the Transverse and Conjugate Diameters given, *viz.* TS and Nn, placed at *Right-angles* in their *Middles*, As before. Then from either End of the Conjugate, *viz.* N (or n) set off half the Transverse Diameter to x. That is, make $Nx = TC$, *(continuing the* Conjugate Nn *when 'tis shorter than* TC*)* Or, which is all one, make $Cx = TC - NC$. Then take any Point in the Line Cx at pleasure: Suppose it at G; and from that Point at G set off the Distance Cx to the Transverse (as at E) *viz.* make $GE = Cx$, and joyn the Points GE with a Right-line, produced so far beyond E as to make $EB = NC$. Consequently $GB = TC$.



Then, I say, where-ever the Point G was taken between C and x, the Point B will just touch *(or fall in)* the *Ellipsis*'s *Periphery*.

Demonstration.

Draw the Right-line BA Perpendicular to TS, *viz.* let BA be a *Semi-ordinate rightly apply'd* to the Transverse Diameter TS; Then $\triangle GCE$ and $\triangle BAE$ will be alike.

Consequent	1	$CE : AE :: EG : EB$. By Theorem 13.
1, And	2	$CE + AE : AE :: EG + EB : EB$. See *p.* 192
But	3	$CE + AE = CA$. $EG \times EB = TC$. And $EB = NC$
Therefore	4	$CA : AE :: TC : NC$
6, in □'s	5	$\square CA : \square AE :: \square TC : \square NC$
5, ∴	6	$\frac{\square CA \times \square NC}{\square TC} = \square AE$
But	7	$\square NC - \square AB = \square AE$
That is,		$\square EB - \square AB = \square AE$
6, 7	8	$\frac{\square CA \times \square NC}{\square TC} = \square NC - \square AB$
8 × □ TC	9	$\square CA \times \square NC = \square NC \times \square TC : - \square AB \times \square TC$
9 ±	10	$\square NC \times \square TC : - \square CA \times \square NC = \square AB \times \square TC$
10, *Analogy*	11	$\square TC : \square NC :: \square TC - \square CA : \square AB$
That is,	12	$TC \times CS : \square NC :: TC \times CA \times TC - CA : \square AB$

Which is according to the common *Properties* of the *Ellipsis*. Therefore the Point B is truly found. Q. E. D.

Hence

Hence it follows, that if a convenient number of such Lines as *G E B* be so drawn (as above directed) from the like number of Points taken between *C* and *x*, &c. their Extream Points (as at *B*) will be those Points by which (*with an even Hand*) the *Ellipsis* may be truly describ'd; as before.

But, if this be well understood, 'twill be *very easie* to conceive how to *describe* an *Ellipsis* very readily, without drawing those Lines, by having a *thin*, *streight*, *narrow Ruler* just the length of *T C*, made somewhat *sharp* at both Ends, upon which, from one of its Ends, set *off* the length of *N C*. Then, if the Point upon the Ruler which represents *E* be gradually or easily moved along the *Transverse T S*, and at the same time the Point or End representing *G* be kept sliding close along the Conjugate *N n*, 'tis evident from the Work above, that the End of the Ruler representing *B* will, by that motion, assign the true *Periphery* of the *Ellipsis* requir'd; for by that motion the *streight Edge* of the Ruler doth supply an infinite number of the aforesaid Lines; as will appear very plain and easie in Practice.

Scholium.

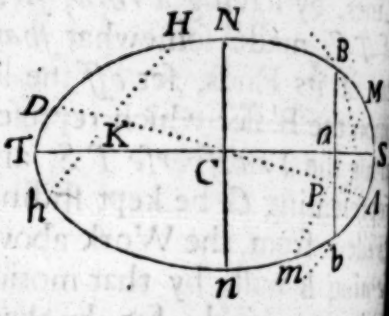
Now, from hence was deduc'd the *first Invention* of that well-contriv'd *Instrument* for drawing an *Ellipsis* by one Motion, commonly call'd the *Elliptical Compasses*, being usually made of Brass, and compos'd of *Three parts*, *Two* of which represent (*or rather apply*) the Transverse and Conjugate Diameters set together at *right Angles*; and the *Third part* is a *Moveable Ruler*, which performs the Office of the last-mention'd *thin* Ruler. But because the making of it is so well known to most Mathematical Instrument-makers, *Especially to that Accurate and Ingenious Artist* Mr. JOHN ROWLEY, *Mathematical Instrument-maker, in* St. Dunstan's Church *in* Fleet-street, London; *who for his great* Skill *in contriving, framing, and graduating* all kind *of Mathematical Instruments, may, I believe, be justly call'd One of the best Workmen of his Trade in* Europe. I think it needless therefore to give a *particular* Description of that *Instrument*.

Also from hence came that *Ingenious Invention* of making *Engines* for *Turning* all sorts of Elliptical or Oval Work, as Oval Frames, Picture-Frames, *&c.*

Sect.

Sect. 5. *Any Ellipsis being given, To find its* Transverse *and* Conjugate *Diameters.*

Suppose the given *Ellipsis* to be *T N S n (in the annexed Scheme)* in which let it be requir'd to find the Transverse Diameter *T S* and its Conjugate *N n.* Draw within the *Ellipsis* any *Two Right Lines* parallel to each other, as *H h* and *M m*, and bisect those Lines, *viz.* find the Middle Point of each, as at *K* and *P*; Then thro' those Points *K* and *P* draw a Right-line, as *D A*, and it will be a Diameter; for it will divide the Ellipsis into two equal Parts, [*See* Defin. 1, *page* 357.]



Consequently the Middle of *D A* will be the true Middle or common Centre of the Ellipsis, as at *C.*

For 'tis the Nature or Property of all Diameters, *howsoever they are drawn in any* Ellipsis, *(as 'tis in a* Circle*) to cut or cross one another in the common Centre or Middle of the Figure*; *as at C.*

Upon the Point *C* describe an *Arch* of any Circle that will cut the Ellipsis Periphery in *Two Points*, as at *B* and *b*; then join those Points *B b* with a *Right Line*, and it will be an *Ordinate* through whose Middle *(as at a)* and the common Centre *C*, the Transverse Diameter *T S* must pass.

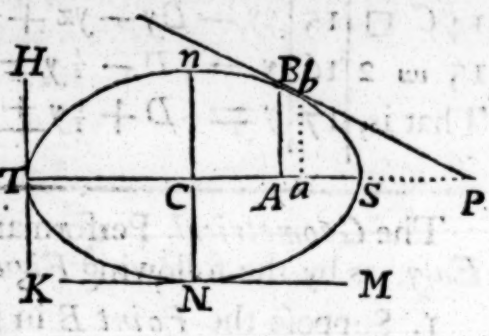
For *B S = S b*, and *B a* is at Right-angles with *T S*; therefore the Line *B b* is an *Ordinate rightly apply'd* to *T S* the Transverse Diameter. And if thro' the Point *C* there be drawn the Right line *N n* parallel to *B b*, it will become the Conjugate; As was requir'd.

Sect. 6. *To draw a* Tangent, *or Right Line, that may touch the Ellipsis Periphery in any Assign'd Point.*

The Drawing of *Tangents* to or from any *assign'd Point* in the Ellipsis Periphery, admits of Three Cases.

Case 1. If it be requir'd to draw a *Tangent* that may touch the Ellipsis in either of the Extream Points of its Transverse Diameter, as at *T* or *S*, it is plain the *Tangent* must be drawn parallel to the Conjugate Diameter *N n*, as *H K* in the following *Figure* is suppos'd to be.

Caſe 2. Or if the *Tangent* muſt be *Drawn* to *touch* the *Ellipſis* in either of the *Extream Points* of its *Conjugate Diameter*, as at *N* or *n*, 'tis as evident that it muſt be *Drawn parallel* to the *Tranſverſe Diameter T S*, as *K M*. Conſequently, if that *Tangent*, and the *Tranſverſe* were both *Infinitely continu'd*, they would never *meet*.



Caſe 3. But if it be requir'd to draw a *Tangent* that may *touch* the *Ellipſis* in any other *Point*, as at *B*, &c.

Then if the *Tangent* and the *Tranſverſe Diameter* be both continu'd, they will *Meet* in ſome *Point*, as at *P*; and thoſe *Two Points* (viz. *B* and *P*) do ſo mutually depend upon each other, that one of them muſt be *Aſſign'd* in order to find the other, that ſo the *Tangent* may by them be truly *Drawn*.

Let $D = TS$. $y = AS$: And $z = AP$. Then if y be given, z may be found by this *Theorem* $\left\{\frac{Dy - yy}{\frac{1}{2}D - y} = z\right.$

Or if z be given, y may be found by this *Theorem*:

Theorem. $\left\{\frac{D + z}{2} \pm \sqrt{\left\{\frac{DD + zz}{4} = y\right.\right.$

Demonſtration.

Draw the *Semi-ordinate b a*, as in the *Figure*; then will $\triangle BAP$ and $\triangle baP$ be alike. Put $x = Aa$ the *Diſtance* between the *Two Semi-ordinates* (viz. *between B A* and *b a*) which we ſuppoſe *Infinitely ſmall*.

Then	1	$z : z - x :: BA : ba$. By *Theorem* 13
But	2	$D - y \times y : D - y + x \times y - x :: \square BA : \square ba$
That is,	3	$Dy - yy : Dy - yy + 2yx - Dx - xx :: \square BA : \square ba$
in □'s	4	$zz : zz - 2zx + xx :: \square BA : \square ba$
Suppoſe	5	$x = 0$ That ſo x may be every where *rejected*
Then	6	$Dy - yy : Dy - yy + 2y - D :: \square BA : \square ba$
And	7	$zz : zz - 2z :: \square BA : \square ba$
6, 7	8	$Dy - yy : Dy - yy + 2y - D :: zz : zz - 2z$.
8 .	9	$2yzz - Dzz = 2yyz - 2Dyz$
÷ 22	10	$yz - \frac{1}{2}Dz = yy - Dy$
10 ±	11	$\frac{1}{2}Dz - yz = Dy - yy$

11 —	12	$z = \frac{Dy - yy}{\frac{1}{2}D - y}$	Which is the 1st *Theorem*, and gives the following *Analogy*.
Analogy	13	$\frac{1}{2}D - y : y :: D - y : z$.	Viz. $CA : SA : TA : AP$
10 — yz	14	$yy - Dy - yz = -\frac{1}{2}Dz$	
14 C □	15	$yy - Dy - yz + \frac{1}{4}DD - \frac{1}{2}Dz + \frac{1}{4}zz = \frac{1}{4}DD + \frac{1}{4}zz$	
15 ww 2	16	$y - \frac{1}{2}D - \frac{1}{2}y = \sqrt{\frac{1}{4}DD + \frac{1}{4}zz}$	
That is,	17	$y = \frac{1}{2}D + \frac{1}{2}y \pm \sqrt{\frac{1}{4}DD + \frac{1}{4}zz}$	Which is the 2d *Theor.* Q.E.D.

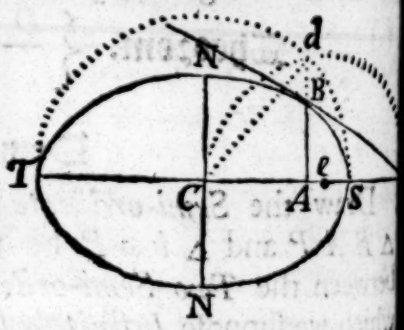
The *Geometrical* Performance of these two *Theorems* is very *Easy*, as by the following *Figure*.

1. Suppose the *Point B* in the *Ellipsis Periphery* were given, and it were requir'd to find the *Point P*, &c.

Make *TC Radius*, and upon the *common Center C Describe* the *Semi-circle Td S*, and *joyn* the *Points C* and *d* with a *Right Line*; then *Bisect* that *Line*, (by *Prob.* 2. *Pag.* 287) and mark the *Point* where the *Bisecting Line* would *Cross* the *Transverse*, as at || *e*. Upon that *Point* || *e*, with the *Radius Ce* (or *Cd*) *Describe* another *Semi-circle*, producing the *Transverse Diameter* to its *Periphery*, and it will *assign* the *Point P*.

For if $D = TS$. $y = AS$. $z = AP$. as before.

Then	1	$D - y \times y = \square dA$	
And	2	$\frac{1}{2}D - y \times z = \square dA$	
For	3	$TA : dA :: dA : SA$	
And	4	$CA : dA :: dA : AP$	
But		$CA = \frac{1}{2}D - y$, &c.	
1, 2	5	$\frac{1}{2}Dz - yz = Dy - yy$ as at the 11 *Step* before	

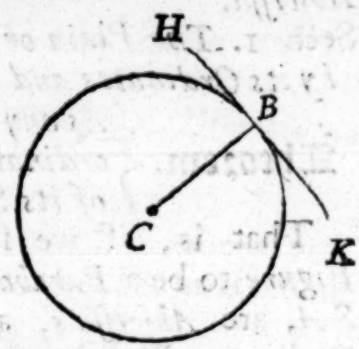


Therefore the *Point P* is truly found. *Consequently*, if a *Right Line* be drawn through those *Points B* and *P*, it will be the *Tangent* requir'd, according to the First *Theorem*.

2. The *Converse* of this is as *Easy*, to wit, if the *Point P* be given, thence to find the *Point B* in the *Ellipsis Periphery*. Thus, *Circumscribe half* the *Ellipsis* with the *Semi-circle Td S*, As before; and *Bisect* the *Distance* between the *Points C* and *P* as at || *e*, *viz.* Let $Ce = cP$. Then making *C e Radius*, upon the *Point* || *c* || *describe* the *Semi-circle C d P*; and from the *Point* where the *Two Semi-circles* intersect or cross each other as at || *d*, draw the *Right Line dA perpendicular* to the *Transverse* *TS*

TS, and it willl *Aſſign* the *Point* of *Contact B* in the *Ellipſis Periphery* through which the *Tangent* muſt paſs.

But a *Practical Method* of drawing *Tangents* to any *aſſign'd Point* in the *Ellipſis Periphery*, may (*without finding the aforeſaid Point P*) be *Eaſily* deduc'd from the following *Property* of *Tangents* Drawn to a *Circle*, which is this.



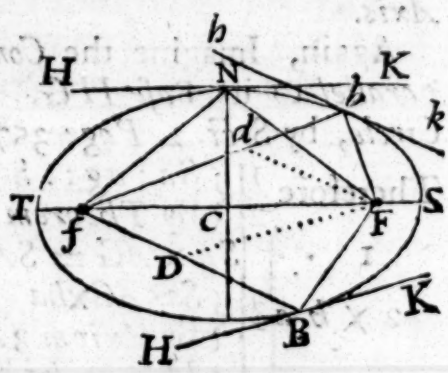
If to any *Radius* of a *Circle*, as *C B*, there be drawn a *Tangent Line* (*as HK*) to *Touch* the *Radius* at the *Point B*, the *Two Angles* which the *Tangent* makes with the *Radius*, will always be *Two Right Angles* (16, 17, 18, 19 *Euclid.* 3.)

That is, $\angle HBC = \angle CBK = 90°$.

In like Manner the *Two Angles* made between the *Tangent* and the *Two Lines* drawn from the *Focus*'s of any *Ellipſis* to the *Point* of *Contact*, will always be *Equal*, but not *Right Angles*, ſave only at the *Two Ends* of the *Tranſverſe Diameter*.

Theſe being well conſider'd, and compar'd with what hath been ſaid in *Page* 366, it muſt needs be *Eaſy* to underſtand the following Way of drawing *Tangents* to any *Aſſign'd Point* in the *Ellipſis Periphery*; which is thus:

Having by the *Tranſverſe* and *Conjugate Diameters* found the *Two Focus's f* and *F*, by *Sect.* 3. from them draw *Two Right Lines* to meet each other in the *Aſſign'd Point* of *Contact*, as *f b* and *F b* (or *f B* and *F B*) in the *Annex'd Figure*. Next ſet *Off* (viz. *make*) $b\,d = b\,F$, (*or* $B\,D = B\,F$) and *joyn* the *Points F d* (or *F D*) with a *Right Line*.



Then, I ſay, if a *Right Line* be drawn through the *Point* of *Contact b* (*or B*) *parallel* to *d F*, *or* (*D F*) it will be the *Tangent* requir'd.

For it is plain, that as the $\angle f\,N\,H = \angle F\,N\,K$ when the *Tangent* is *parallel* to the *Tranſverſe Diameter*, even ſo is the $\angle f\,b\,h = \angle F\,b\,k$, (*and* $\angle f\,B\,H = \angle F\,B\,K$) and will be every where ſo, as the *Point* of *Contact b* (*or B*) and its *Tangent* is carry'd about the *Ellipſis Periphery* with the *Lines f b F* (*or f B F*).

CHAP. III.

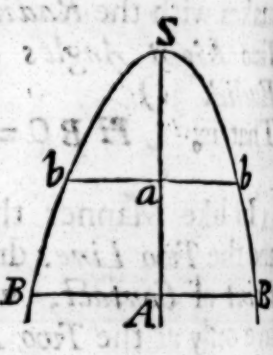
Concerning the Chief Properties of every Parabola.

NOte, *In every Parabola, the intercepted Diameter, or that Part of its Axis which is between the Vertex and that Ordinate which limits its Length, As S a or S A, &c. is call'd Abscissa.*

Sect. 1. *The Plain or Figure of every Parabola, is proportion'd by its Ordinates and Abscissa's, as in the following Theorem.*

Theorem. *As any one Abscissa: is to the Square of its Semi-ordinate :: so is any other Abscissa: to the Square of its Semi-ordinate.*

That is, if we suppose the *annex'd Figure* to be a *Parabola*, wherein *Sa*, and *S A*, are *Abscissa's*, and *b a b*, *B A B*, *Ordinates Rightly* apply'd, it will be *Sa*: □*ba* :: *SA*: □*BA* } whereſoever
Or *Sa*: *SA*:: □ *ba*: □*BA* } the *Points a*, *A*, are taken,
And so for any other *Abcissa*, &c.



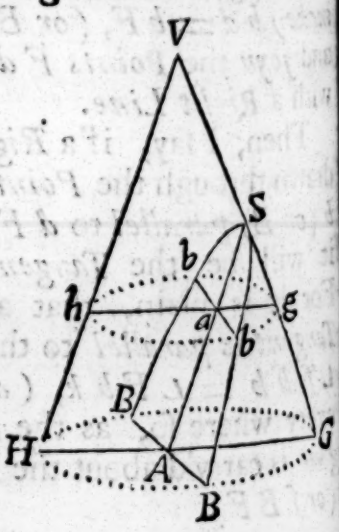
Demonstration.

Let the following *Figure H V G* represent a *Right Cone* cut into *Two* Parts by the *Right Line S A*, *parallel* to its Side *V H*. Then the *Plain* of that *Section*, viz. *B b S b B* will be a *Parabola*, by *Sect.* 4. *Page* 358, wherein let us suppose *S A* to be its *Axis*, and *b a b*, *B A B* to be *Ordinates rightly* Apply'd to that *Axis*.

Again, Imagine the *Cone* to be *Cut* by the *Right Line h g parallel* to its *Base H G*. Then will *h g* be the *Diameter* of a *Circle*, by *Sect.* 2. *Page* 357. And △ *S a g* like to △ *S A G*.

Therefore 1 { *Sa* : *ag* :: *S A* : *AG*
2 { By *Theorem* 13.
1 ... *Sa* × *AG* = *SA* × *ag*
2 × *ba* 3 { *Sa* × *AG* × *ba* = *SA* × *ag* × *ba*
{ By *Axiom* 3.
But 4 { *H A* = *ba* because *S A*
{ is *parallel* to *V H*
And { 5 □ *BA* = *AG* × *HA* } By Lemma
□ *ba* = *ag* × *ha* } P. 357.
3, 4, 5 { *SA* × □ *BA* = *SA* × □ *ba*
{ By *Axiom* 5.
6, *Analogy* 7 { *Sa* : □ *ba* :: *SA* : □ *BA*
{ Vide *Page* 194.

Q. E. D.



These

These *Proportions* being prov'd to be the *Common Property* of every *Parabola*, all that is farther requir'd about that *Section*, or *Figure*, may from thence eaſily be deduc'd.

Sect. 2. *To find the* Latus Rectum, *or Right* Parameter *of any* Parabola.

The *Latus Rectum* of a *Parabola* hath the ſame *Ratio* or *Proportion* to any *Abſciſſa*, and its *Semi-ordinate*, as the *Latus Rectum* of any *Ellipſis* hath to its *Tranſverſe* and *Conjugate Diameters*, and may be found by this *Theorem*.

Theorem { *As any Abſciſſa : is in Proportion to its Semi-ordinate :: ſo is that Semi-ordinate : to the* Latus Rectum.

Let L = the *Latus Rectum*.

Then	1	$S\,a : b\,a :: b\,a : L$	{ where-ever the *Points a* and
And	2	$S\,A : B\,A :: B\,A : L$	{ *A*, are taken in the *Axis*.
1 ∴	3	$\frac{\square\, b\,a}{S\,a} = L$: Or $S\,a \times L = \square\, b\,a$	
2 ∴	4	$\frac{\square\, B\,a}{S\,a} = L$: Or $S\,A \times L = \square\, B\,A$	
3 = 4	5	$\frac{\square\, B\,A}{S\,A} = \frac{\square\, S\,a}{b\,a}$ Per *Axiom* 5.	
5 ×		$S\,a \times \square\, B\,A = S\,A \times \square\, b\,a$	Which gives this
Analogy	6	$S\,a : \square\, b\,a :: S\,A : \square\, B\,A$	The ſame as at the 7th

Step of the laſt *Proceſs*; therefore L (*thus found*) is the true *Latus Rectum*, by which all the *Ordinates* may be *Regulated* and found, according to its *Definition* in *Section* 4. *Page* 358.

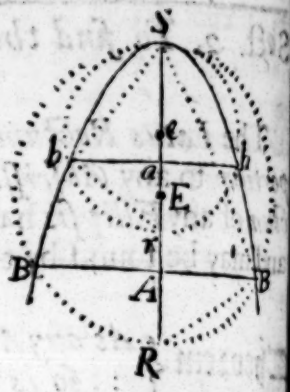
For by the *Third Step* $S\,a \times L = \square\, b\,a$. And by the 4th *Step* $S\,A \times L = \square\, B\,A$. *Conſequently* $\sqrt{S\,a \times L} = b\,a$ and $\sqrt{S\,A \times L} = \square\, B\,A$; and for any other *Ordinate*.

Or if the *Ordinates* are given, to find their *Abciſſa's*; then it will be, $L : b\,a :: b\,a : S\,a$. And $L : B\,A :: B\,A : S\,A$, &c. *Conſequently* $\frac{\square\, b\,a}{L} = S\,a$. And $\frac{\square\, B\,A}{L} = S\,A$, &c.

From the Conſideration of theſe *Proportions*, it will be *eaſy* to conceive how to find the *Latus Rectum Geometrically*, thus:

Joyn

Joyn the *Vertical Point S* of the *Axis*, and either *Extream Point* of any *Ordinate*, as B, (*or b*) with a *Right Line*, viz. *SB* (*or Sb*) and *Biſect* that *Line*, (by *Problem* 2. *Page* 287.) marking the *Point* where the *biſecting Line* doth *Interſect* or *croſs* the *Axis*, as at *E*, (*or e*) and with the *Radius SE* (*or Se*) upon the *Point E*, (*or e*) *Deſcribe a Circle*; (*as in the Annex'd Figure*) then will the Diſtance between the *Ordinate* and that *Point* where the *Circle's Periphery* cuts the *Axis*, viz. *AR* (*ro ar*) be the true *Latus Rectum* requir'd.



For $SA : BA :: BA : AR$. And $Sa : ba :: ba : ar$. By *Theor.* 13. Therefore $AR = L$. And $ar = L$ by the 1*ſt* and 2*d Steps* above.

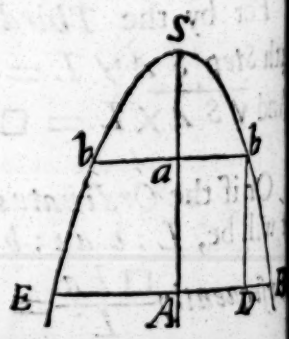
Conſectary.

From theſe *Proportions* of finding the *Latus Rectum*, it will be eaſy to deduce and *Demonſtrate* this following *Theorem.*

Theorem { *As the* Latus Rectum : *Is to the Sum of any Two Semi-ordinates* :: *ſo is the Difference of thoſe Two Semi-ordinates* : *to the Difference of their Abſciſſa's.*

Suppoſe any *Right Line Drawn* within the *Parabola*, as *b D*, *parallel* to its *Axis SA*; then will that *Line* (viz. *bD*) be a *Diameter* (by *Def.* 5. *Pag.* 359) which will make $ED = AB + ab$, $DB = AB - ab$, and $bD = SA - Sa$. Then it will be $L : ED :: DB : bD$, according to the *Theorem.*

Demonſtration.



First	1	$SA = \frac{\square BA}{L}$ By *Step* 2 of the laſt *Proceſs.*
And	2	$Sa = \frac{\square ba}{L}$ By *Step.* 1. of the laſt *Proceſs.*
1 — 2	3	$SA - Sa = \frac{\square BA - \square ba}{L}$
3 × L	4	$\overline{SA - Sa} \times L = \square BA - \square ba$
But	5	$\square BA - \square ba = \overline{BA + ba} \times \overline{BA - ba}$
4, = 5, 6	6	$\overline{SA - Sa} \times L = \overline{BA + ba} \times \overline{BA - ba}$
6, *Analogy*	7	$L : BA \times ba :: BA - ba : SA - Sa$
Or	8	$L : ED :: DB : bD$

{ Which gives the following *Analogy.*

This peculiar Property of the Parabola was first publish'd Anno 1684, *by one Mr.* Thomas Baker, *Rector of* Bilhop Nympton *in* Devonshire, *in a Treatise intituled,* The Geometrical Key: Or, The Gate of Æquations unlock'd; *wherein he hath shew'd the Geometrical Construction and Solution of all Cubick and Biquadrantick Adfected Æquations by one general Method, which he calls a* Central Rule, *Deduc'd from this peculiar Property of the Parabola.*

Sect. 3. *To find the* Focus *of any Parabola.*

The *Focus* of every *Parabola* is that *Point* in its *Axis* through which *the Latus Rectum* doth pass. (See *Definition* 3. *Sect.* 4. *Page* 359.) Therefore its Distance from the *Vertex* of the *Parabola* may be easily found, either by the *Latus Rectum* it self, or by any other *Ordinate*, and its *Abscissa*.

Thus, suppose the *Point* at *F* to be the *Focus*, *S* the *Vertex*, the *Ordinate* $RFR = L$ the *Latus Rectum*, and *b a b* any other *Ordinate*.

Then will $SF = \frac{1}{4}L$. Or $SF = \frac{\square ba}{4\,Sa}$

Demonstration.

First	1	$SF \times L = \square FR$. By *Sect.* 2. *Page* 375.
And	2	$FR = L$. For the Ordinate $RFR = L$ as above
2 & 2	3	$\square FR = \frac{1}{4}\square L = \frac{1}{2}L \times \frac{1}{2}L$
4, = 3	4	$SF + L = \frac{1}{4}\square L$
4 ÷ L	5	$SF = \frac{1}{4}L$ As by *Definition* 4. *Sect.* 4. *Page* 359.
Again	6	$\frac{\square ba}{Sa} = L$ by the *Third Step* in Page 375.
Conseq.	7	$\frac{\square ba}{4\,Sa} = \frac{1}{4}L$. &c. As above. Q. E. D.

Sect. 4. *To* Describe, *or Draw a Parabola several Ways.*

Note, *There are Two or Three Ways of Drawing a Parabola Instrumentally at one Motion; but because those Instruments or Machines are not only too perplex'd for a Learner to manage, but also a little subject to Error, I have therefore chosen to shew how that Figure may be (the best) Drawn by a convenient Number of Points,* viz. *Ordinates found, either Numerically or Geometrically, according to the* Data; *which, if the Work of the Three last Sections be well consider'd, must needs be very Easy.*

1. If

1. If any *Ordinate* and its *Abscissa* are given, there may by them be found as many *Ordinates* as you please to assign or take *Points* in the *Parabola's Axis*, (by *Sect.* 1. *Page* 374.) and the *Curve* of the *Parabola* may be drawn by the *Extream Points* of those *Ordinates*, as the *Ellipsis* was *Page* 367.

2. If the *Latus Rectum*, and either any *Ordinate*, or its *Abscissa*, are given, then any assign'd *Number* of *Ordinates* may by them be found, (by *Sect.* 2. *Page* 375.) either *Numerically* or *Geometrically*, &c.

3. If only the *Distance* of the *Focus* from the *Vertex* of the *Parabola* be given, any assign'd *Number* of *Ordinates* may be found by it. For $SF = \frac{1}{4}L$ the *Latus Rectum*, and $\frac{1}{2}L = FR$ as in the *Last Section*; and it will be, as SF: is to $\square FR$:: so is any other *Abscissa* viz. (*Sa*, or *SA*, &c.): to the *Square* of its *Semiordinate*, (viz. $\square ba$, or $\square BA$) according to the common *Property* of the *Parabola*.

Altho' any of these Ways of finding the *Ordinates* are easy enough, yet that Way which may be deduc'd from the following *Proposition* will be found much more Easy, and ready in *Practice*.

Proposition. *The Sum of any Abscissa and focal Distance from the Vertex, will be Equal to the Distance from the Focus to the Extream Point of the Ordinate which cuts off that Abscissa.*

For Instance, suppose S to be the *Vertex* of any *Parabola*, the *Point* F to be its *Focus*, and AB any *Semi-ordinate Rightly Apply'd* to its *Axis* SA. Then I say, where-ever the *Point* A is taken in the *Axis*, it will be $SA + SF = FB$. Consequently, if $Sf = SF$, it will be $fA = FB$.

Demonstration

First	1	$SF = \frac{1}{4}L$ by the 7th *Step*, *Sect.* 3.
Ergo	2	$fA = FA + \frac{1}{2}L$ by *Construction* above.
2 @ 2	3	$\square fA = \square FA + \overline{FA \times L} : \frac{1}{4}LL$
Again	4	$SA = FA + \frac{1}{4}L$ by the *Supposition* and *Figure*.
4 × L	5	$SA \times L = \overline{FA \times L} : + \frac{1}{4}LL$ But $SA \times L = \square AB$
Ergo	6	$\square AB = \overline{FA \times L} : + \frac{1}{4}LL$
3 — 6	7	$\square fA - \square AB = \square FA$. Conse. $\square fA = \square FA + \square AB$
But	8	$\square FA + \square AB = \square FB$. By *Theorem* 11.
Ergo	9	$\square fa = \square FB$
9 ww 2	10	$fA = FB$ Q. E. D.

This

This *Proposition* being well understood, 'twill be very easily ap-ly'd to *Practice*, supposing the *Focal Distance* given, or any other *Data* by which it may be found.

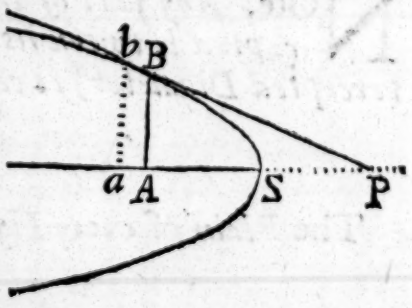
Thus draw any *Right-line* to represent the *Parabola*'s *Axis*, nd from its *Vertical Point*, as at *S*, set *Off* the *Focal Distance* oth upwards and downwards, *viz.* make $Sf = SF$, the Distance f the given *Focus* from the *Vertex*; as in the *Scheme*: Then by he *Proposition* 'tis evident, that if never so many Lines be drawn *rdinately* at *Right-angles* to the *Axis*, the true Distance between he Point *f* out of the *Parabola*, and any of those Lines (or *Ordi-ates*) being measur'd or set off from the *Focus F* to the same ine or Ordinate, 'twill assign the true Point in that Line through hich the *Curve* must pass; that is, it will shew the true Limits Length of that *Ordinate*; as at *B* in the last Scheme.

Proceeding on in the very same manner from *Ordinate* to *Ordi-ate*, you may with great Expedition and Exactness find as many *rdinates (or rather their Points only, like B)* as may be thought nvenient, which being all joyn'd together with an even Hand, ill form the *Parabola* requir'd.

NB. *The more* Ordinates *(or their* Points*) there are found, nd the nearer they are to one another, the easier and exacter ay the* Curve *of the* Parabola *be drawn. The same is to be ob-erv'd when any other* Curve *is requir'd to be drawn by* Points.

Sect. 5. *To draw a* Tangent *to any given* Point *in the* Curve *of a* Parabola.

Tangents are very easily drawn to the *Curve* of any *Parabola*; or, supposing *S* to be its *Vertex*, *B* he Point of *Contact*, (*viz.* the Point here the *Tangent* must touch the *urve*) and *P* the Point where the *angent* will intersect *(or meet with)* he *Parabola's Axis* produced: Then, from the Point of *Contact B* there e drawn the *Semi-ordinate B A* at ight-angles to the Axis *S A*, where-ever the Point *A* falls in the Axis, 'twill be $SP = SA$.



Demonstration.

Draw the *Semi-ordinate b a (as in the Figure)* then will the $\triangle BAP$ and $\triangle baP$ be alike. Let $y = AS$ the *Abscissa*,

and $z = SP$; put $x = Aa$ the Distance between the two *Semi-ordinates*, which we suppose to be infinitely *near* each other, as in the *Ellipsis*, page 371.

Then	1	$y + z : BA :: y + z + x : ba$. Per *Theorem* 13
1, Or	2	$y + z : yx + xx :: BA : ba$. See *Page* 192.
Again	3	$y : \square BA :: y + x : \square ba$. Per *Theorem* page 374
3, Or	4	$y : y + x :: \square BA : \square ba$
2 in □'s	5	$yy + 2yz + zz : yy + 2yz + 2yz + zz + 2zx + xx :: \square BA : \square ba$
4, 5	6	$y : y + x :: yy + 2yz + zz : yy + 2yz + 2yx + zz + 2zx + xx$
6 ∴	7	$yy + 2yz + yx + zz + 2zx + \frac{zzx}{y} = yy + 2yz + 2yx + zz + 2zx + xx$.
That is,	8	$\frac{zzx}{y} = yx + xx$ Consequently $\frac{zz}{y} = y + x$
Suppose	9	$x = 0$ And *Rejected*, As in the *Ellipsis*, page 371
Then	10	$\frac{zz}{y} = y$ Consequently $zz = yy$
10 2	11	$z = y$ That is, $SP = SA$

Q. E. D.

CHAP. IV.

Concerning the chief Properties of the Hyperbola.

NOte, *Any part of the* Axis *of an* Hyperbola, *which is intercepted between its Vertex and any Ordinate*, (viz. *any intercepted Diameter*) *is call'd an* Abscissa; *as in the* Parabola.

Sect. 1.

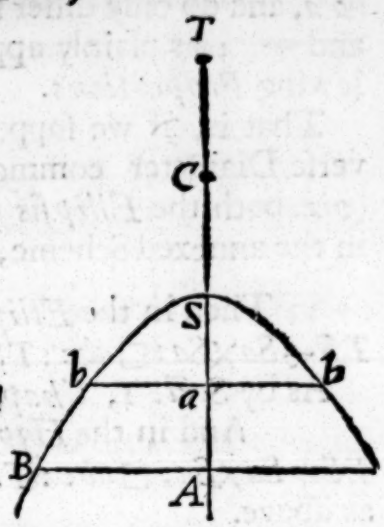
The *Plain* of every *Hyperbola* is proportion'd by this general *Theorem.*

Theorem. *As the Sum of the Transverse and any* Abscissa *multiply'd into that* Abscissa : *is to the Square of its Semi-ordinate* : : *so is the Sum of the Transverse and any other* Abscissa *multiply'd into that* Abscissa *to the Square of its Semi-ordinate.*

That is, if TS be the Transverse Diameter,

And $\begin{cases} Sa, SA \text{ Abscissa's.} \\ ba, BA \text{ Semi-ordinates.} \end{cases}$

Then is $\begin{cases} Ta = TS + Sa \\ TA = TS + SA \end{cases}$



And it will be

$$Ta \times Sa : \square ba :: TA \times SA : \square BA.$$

That is,

$$TS + Sa \times Sa : \square ba :: TS + SA \times SA : \square BA$$

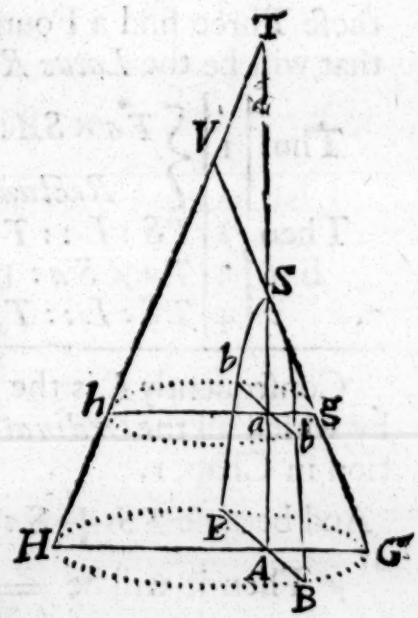
&c.

Demonstration.

Let the following Figure HVG represent a *Right Cone* cut into two parts by the Right-line SA; then will the Plain of that Section be an *Hyperbola*, (by *Sect.* 5, *Chap.* 1.) in which let SA be its Axis, or intercepted Diameter, bab and BAB *Ordinates rightly apply'd (as before in the* Parabola) and TS its Transverse Diameter.

Again, if the Cone is suppos'd to be *cut* by hg, parallel to its Base HG, it will also be the Diameter of a Circle, *&c.* as in the *Ellipsis* and *Parabola*.

Then will the $\triangle Sga$ and $\triangle SGA$ be like, also the $\triangle Tah$ and TAH will be like; therefore it

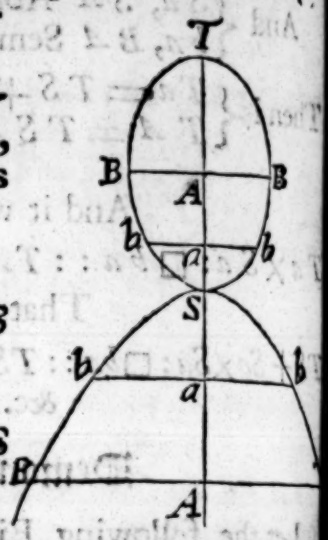


will be	1	$Sa : ag :: SA : AG$
And	2	$Ta : ah :: TA : AH$
1 ∵	3	$Sa \times AG = SA \times ag$
2 ∵	4	$Ta \times AH = Ta \times ah$
3 × 4	5	$Sa \times Ta \times AG \times AH = SA \times TA \times ag \times ah$
But	6	$ag \times ah = \square ab$
And	7	$AG \times AH = \square AB$ per *Lemma* page 357.
5, 6, 7	8	$Sa \times Ta \times \square AB = SA \times TA \times \square ab$ which give the following
8, *Anal.*	9	$Sa \times Ta : \square ab :: SA \times TA : \square AB$, &c.

Q. E. D.

These *Proportions* are the common Property of every *Hyperbola*, and do only differ from those of the *Ellipsis* in the *Sines* + and — ; as plainly appears in all the following *Proportions*.

That is, if we suppose *TS* the Transverse Diameter common to both Sections, (*viz.* both the *Ellipsis* and *Hyperbola*) as in the annexed Scheme,



Then in the *Ellipsis* it will be
$TS - Sa \times Sa : \square ab :: TS - SA \times SA : \square AB$
As by *Sect.* 1, *Chap.* 2.

And in the *Hyperbola* it is
$TS + Sa \times Sa : \square ab :: TS + SA \times SA : \square AB$
as above.

And therefore all that is farther requir'd in the *Hyperbola*, may (*in a manner*) be found as in the *Ellipsis*, due Regard being had to *changing* of the *Signs*.

Sect. 2. *To find the* Latus Rectum, *or* Right Parameter, *of any Hyperbola.*

From the last Proportion take either of the Antecedents and its Consequent, *viz.* either $Ta \times SA : \square ab$. Or $TA \times SA : \square AB$. To them bring in the Transverse *TS* for a *third Term*, and by those Three find a Fourth Proportional (*as in the Ellipsis*) and that will be the *Latus Rectum*.

Thus	1	$Ta \times SA : \square ab :: TS : \frac{\square ab \times TS}{Ta \times SA}$ = the *Latus Rectum*, which call *L* (*as in the* Parabola.)
Then	2	$TS : L :: Ta \times Sa : \square ab$.
But	3	$Ta \times Sa : \square ab :: TA \times SA : \square AB$ Therefore
2, 3	4	$TS : L :: TA \times SA : \square AB$, &c.

Consequently *L* is the true *Latus Rectum*, or *right Parameter*, by which all the *Ordinates* may be found, according to its Definition in Chap. 1.

And because $TS + Sa = Ta$, let it be $TS + Sa$ instead of Ta.

Then it will be $\frac{\square ab \times TS}{TS \times SA : + \square Sa} = L$

And in the Ellipsis *it would be* $\frac{\square ab \times TS}{TS \times SA : - \square Sa} = LR = L$

Sect.

Sect. 3. *To find the* Focus *of any Hyperbola.*

The *Focus* being that Point in the *Hyperbola*'s Axis through which the *Latus Rectum* muſt paſs, (*as in the* Ellipſis *and* Parabola) it may be found by this *Theorem*:

Theorem *To the Rectangle made of half the Tranſverſe into half the* Latus Rectum, *add the Square of half the Tranſverſe, the Square Root of that Sum will be the diſtance of the* Focus *from the Centre of the* Hyperbola.

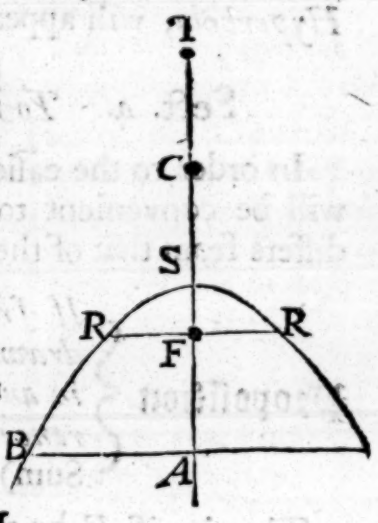
Demonſtration,

Suppoſe the Point at F, in the annex'd Scheme, to be the *Focus* ought; then will $FR = \frac{1}{2}L$.

Let $TC = CS$ be half the Tranſverſe; hen is the *Point C* call'd the *Centre* of the *Hyperbola* (for a Reaſon that ſhall be here-ter ſhew'd.)

Again; Let $CS = d$, And $SF = a$

Then	1	$2d : L :: 2d + a \times a : \frac{1}{4}LL$
hat is,	2	$TS : L :: TS + SF \times FS : \square FR$
1 ∴	3	$\frac{1}{2}dL = 2da + aa$
+ dd	4	$dd + \frac{1}{2}dL = ddd + 2da + aa$
ω 2	5	$\sqrt{dd + \frac{1}{2}dL} = d + a = FC.$
5,—d	6	$\sqrt{dd + \frac{1}{2}dL} - d = a = SF$



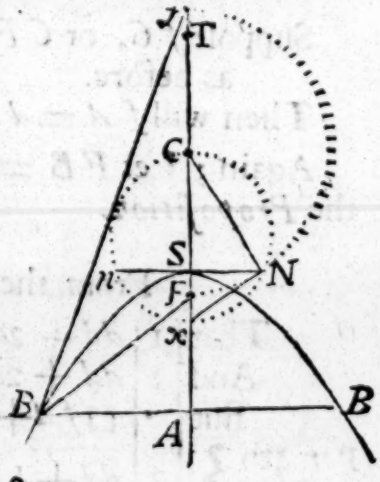
the *Ellipſis* 'tis, $2d : L :: 2d - a \times a : \frac{1}{4}LL$.
That is, $\frac{1}{2}dL = 2da - aa$, &c.

The Geometrical Effection of the laſt *Theorem* is very eaſily rform'd, thus;

Make $Sx = \frac{1}{2}L$, viz. *half* the *Latus Rectum*; and let $CS = d$, above.

Upon Cx *(as a Diameter)* deſcribe a rcle, and at S the *Vertex* of the *Hyperbola* aw the Right-line $n\,S\,N$ at Right-angles Cx; then joyn the *Points* CN with a ght-line, and 'twill be $CN = d + a = FC$.

For	1	$CS : SN :: SN : Sx$ per *Fig.*
hat is,	2	$d : SN :: SN : \frac{1}{2}L$
2 ∴	3	$\frac{1}{2}dL = \square SN$
But	4	$dd + \square SN = \square CN$
3, 4	5	$dd + \frac{1}{2}dL = \square CN$
ω 2	6	$\sqrt{dd + \frac{1}{2}dL} = CN = d + a$, &c.



Now

Now here is not only found the Distance of the *Hyperbola's Focus*, either from its Centre *C*, or Vertex *S*, but here is also found that Right-line usually call'd its *Conjugate Diameter*, viz. the Line *nSN*, which bears the same proportion to the Transverse and *Latus Rectum* of the *Hyperbola* as the Conjugate Diameter of the *Ellipsis* doth to its Transverse and *Latus Rectum*.

For, in the *Ellipsis* $TS : Nn :: Nn : LR$. per Sect. 2, pag. 363.

Consequently $\frac{1}{2}TS : \frac{1}{2}Nn :: \frac{1}{2}Nn : \frac{1}{2}LR$.

But $\frac{1}{2}TS = d$. $\frac{1}{2}Nn = SN$, and $\frac{1}{2}LR = \frac{1}{2}L$.

Therefore $d : SN :: SN : \frac{1}{2}L$. As at the 2d Step above.

What Use the aforesaid Line *nSN* is of, in relation to the *Hyperbola*, will appear farther on.

Sect. 4. *To* Describe *an* Hyperbola *in Plano.*

In order to the easie Describing of an *Hyperbola* in *Plano*, it will be convenient to premise the following *Proposition*, which differs from that of the *Ellipsis* in Sect. 3, Ch. 2, only in the *Sines*.

Proposition. *If from the* Focus's *of any* Hyperbola *there be drawn two Right-lines, so as to meet each other in any Point of the* Hyperbola's *Curve, the Difference of those Lines* (in the *Ellipsis* 'tis their Sum) *will be Equal to the Transverse Diameter.*

That is, if *F* be the *Focus*, and it be made $Cf = CF$, (*as in the last Scheme*) then the Point *f* is said to be a *Focus* out of the Section, (*or rather of the opposite Section*) and it will be $fB - FB = TS$.

Demonstration.

Suppose fC, or $CF = z$, and $SA = x$. Let CS, or $TC = d$, as before.

Then will $fA = d + x + z$, and $FA = d + x - z$.

Again; Let $FB = h$, and $fB = b$. Then $2d = h - b$, by the *Proposition*.

From these *substituted Letters* it follows,

That	1	$dd + 2dx + 2dz + xx + 2zx + zz = \square fA$
And	2	$dd + 2dx - 2dz + xx - 2zx + zz = \square FA$
But		$\square fA + \square AB = \square fB$. And $\square fA + AB = \square FB$
Per 4th of *last*	3	$dd + \frac{1}{2}dL = da + 2da + aa = \square FC = zz$.

3 – *dd*

3 − *dd*	4	$zz - dd = \frac{1}{2}dL$
4 ÷ $\frac{1}{2}d$	5	$\frac{zz - dd}{\frac{1}{2}d} = L$
Again	6	$2d : L :: 2d + x \times x : \square AB$. By *common Properties.*
5, 6	7	$2d : \frac{zz - dd}{\frac{1}{2}d} :: 2dx + xx : \square AB$
7 ∴	8	$\frac{2dzzx + zzxx - 2dddx - ddxx}{dd} = \square AB$
1 + 8	9	$\left\{\begin{array}{l} dd + 2dx + 2dz + xxt + +2zx + zz + \\ \frac{2dzzx + zzxx - 2dx - ddxx}{dd} \end{array}\right\} = \square fA + \square AB = hh$
2 + 8	10	$\left\{\begin{array}{l} dd + 2dx - 2dz + 2x + 2zx + zz + \\ \frac{2dzzx + zzxx - 2d^3x - ddxx}{dd} \end{array}\right\} = \square FA + \square AB = bb$
9 + *d*	11	$d^4 + 2d^3z + 2ddzx + ddzz + 2dzzx + zzxx = ddhh$
10 × *dd*	12	$d^4 - 2d^3z - 2ddzx + ddzz + 2dzzx + zzxx = ddbb$
11 ω 2	13	$dd - dz + zx = dh$
12 ω 2	14	$dd - dz - zx = db$
13 ÷ *d*	15	$d + z + \frac{zx}{d} = h$
14 ÷ *d*	16	$d - z - \frac{zx}{d} = b$
16, or	17	$z + \frac{zx}{d} - d = b$
17 − 17	18	$2d = h - b$

Altho' the Æquation at the 16th Step be in it ſelf impoſſible, becauſe z is greater than d, (by the 4th Step) yet from thence it will be eaſie to conclude, that the Difference between d and $z + \frac{zx}{d}$ *will give the true Value of* b; *as in the 17th Step.*

But becauſe I would leave no room for the Learner to doubt about *changing* the *Æquation*, $d - z - \frac{zx}{d} = b$ into that of $z + \frac{zx}{d} - d = b$, it may be convenient to illuſtrate the whole Proceſs into *Numbers*, whereby *(I preſume)* 'twill plainly appear that $h - b = TS$.

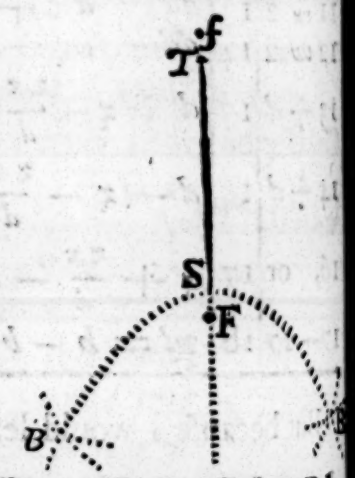
In order to that, Let the **Tranſverſe** $TS = 2d = 12$. Then $d = 6$
Suppoſe the Abſciſſa $SA = x = 4$, And the Semi-ordinate $AB = 3$

Firſt	1	$TS + SA \times SA : \square AB :: TS : L$. *per* Sect. 2.
1, *viz.*	2	$12 + 4 \times 4 = 64 : 9 :: 12 : 1,6875 = L$
Again	3	$\sqrt{dd + \frac{1}{2}dL} = d + a = CF$ *per* Sect. 3.
3, *viz.*	4	$\sqrt{36 + 5,0625} = 6,408 = CF = z$
Then	5	$d + x + z = 6 + 4 + 6,408 = 16,408 = fA$
And	6	$d + x - z = 6 + 4 - 6,408 = 3,592 = FA$

5 & 2	7	$269,2224 = \square fA$
6 & 2	8	$12,9024 = \square FA$
But	9	$9 = \square AB$ For $AB = 3$ by Supposition.
7 + 9	10	$278,2224 = \square fA + \square AB = \square fB$
8 + 9	11	$21,9024 = \square FA + \square AB = \square FB$
10 ww 2	12	$16,68 = fB$
11 ww 2	13	$4,68 = FB$
12 — 13	14	$12,00 = fB - FB = TS$. Which was to be prov'd.

If this *Proposition* be truly understood, it must needs be easie to conceive how to describe the Curve of any *Hyperbola* very readily by *Points*, when the *Transverse Diameter* and the Focus are given, *(or any other Data by which they may be found, as in the precedent Rules)* thus;

Draw any streight Line at pleasure, and on it *set off* the Length of the given *Transverse TS*, and from its Extream Points or Limits, *viz. TS*, set off $Tf = Sf$, the Distance of the given *Focus (viz. the Point f without, and F within the Section, as before)*; that done, upon the *Point f*, *(as a Centre)* with any assum'd *Radius* greater than *TS*, describe an Arch of a Circle; then from that *Radius* take the Transverse *TS*, making their Difference a second *Radius*, with which, upon the Point *F* within the Section, describe another Arch to *cut* or *cross* the first Arch, as at *B*; Then will that Point *B* be in the Curve of the *Hyperbola*, by the *last* Proposition. And therefore 'tis plain, that proceeding on in this manner, you may find as many Points *(like B)* as may be thought convenient, *(the more there are, and nearer they are together, the better)* which being all *joyn'd* together with any even Hand *(as in the* Parabola) will form the *Hyperbola* requir'd.



There are several other Ways of delineating an *Hyperbola* in *Plano*: One Way is, by finding a competent Number of *Ordinates*, as by Section 1, *&c.* but I think none so easie and expeditious as this *Mechanical way:* I shall therefore, for Brevity sake, pass over the rest, and leave them to the Learner's practice, as being easily deduced from what hath been already said.

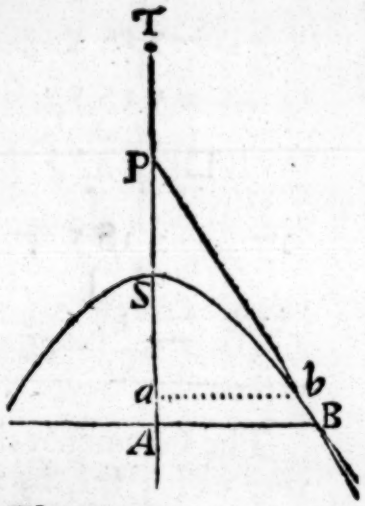
Sect.

Sect. 5. *To draw a* Tangent *to any given Point in the Curve of an* Hyperbola.

The drawing of a *Tangent* that will touch any given Point in the Curve of an *Hyperbola*, may be eaſily perform'd by help of a *Theorem*; as in the *Ellipſis*, Sect. 6, Chap. 2.

Let $\begin{cases} D = TS \text{ the } \textit{Tranſverſe Diameter.} \\ L = \text{the } \textit{Latus Rectum.} \\ y = SA \text{ the } \textit{Abſciſſa.} \end{cases}$

And $z = AP$ { the Diſtance between the *Ordinate* and that Point in the *Tranſverſe* cut by the *Tangent*:



Then if y be given, z may be found by this *Theorem*, $\left\{ \frac{Dy + yy}{\frac{1}{2}D + y} = z \right.$

[*which differs from that in the* Ellipſis *only in Sines*, Vide *page* 371.]

Or, if z be given, then y may be found by this *Theorem*:

Theorem. $\sqrt{\frac{DD + zz}{4}} : + \frac{1}{2}z - \frac{1}{2}Dy$

Demonſtration.

Draw the *Semi-ordinate* $b\,a$, as in the Figure, and put $x = Aa$ { an infinite ſmall Space between the two *Semi-ordinates*; *as before in the* Ellipſis, &c.

Then	1	$D : L :: Dy + yy : \square AB$
That is,	2	$TS : L :: TS + SA \times SA : \square AB$
1 ∴	3	$\frac{DyL + yyL}{D} = \square AB$
Again	4	$D : L :: Dy + yy - 2yx - Dx + xx : \square ab$
That is,	5	$TS : L :: TS + Sa \times Sa : \square ab$
4 ∴	6	$\frac{DyL + yyL - 2yxL - DxL + xxL}{D} = \square ab$
per *Figure*	7	$z : AB :: z - x : ab$, viz. $PA : AB :: Pa : ab$
7 in □'s	8	$zz : \square AB :: zz - 2zx + xx : \square ab$
Suppoſe	9	$x = 0$ and every where rejected (*as in the* Ellipſis)
Then 3, 9	10	$zz : \frac{DyL + yyL}{D} :: zz - 2z : \square ab$
10 ∴	11	$\frac{Dy\,Lzz + yy\,Lzz - 2Dy\,Lz - 2yy\,Lz}{Dzz} = \square ab$

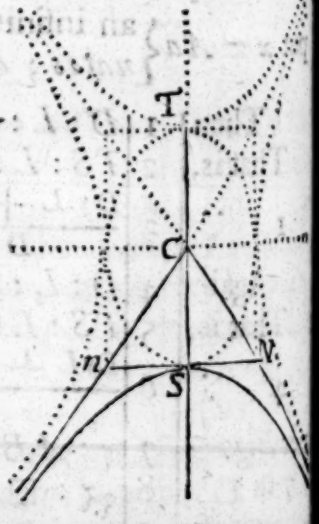
			$\left\{\begin{array}{l}\frac{DyL+yyL-2yL-DL}{D} = \\ \frac{DyLzz+yyLzz-2DyLz-2yyLz}{Dzz}\end{array}\right.$
6,	11	12	
12 reduc'd		13	$\frac{1}{2}Dz+zy=Dy+yy$
13 *Analogy*		14	$\frac{1}{2}D+y:y::D+y:z$, viz. $CA:SA::TA:AP$
13 $\div\frac{1}{2}D-y$		15	$z=\frac{Dy+yy}{\frac{1}{2}D\times y}$ Which is the *first Theorem.*
13 $-zy$		16	$yy+Dy-zy=\frac{1}{2}Dz$
16 $\square G$		17	$yy+Dy-zy+\frac{DD-2Dz+zz}{4}=\frac{DD+zz}{4}$
17 ω	2	18	$y+\frac{1}{2}D-\frac{1}{2}z=\sqrt{\frac{DD+zz}{4}}$
18	$\pm$	19	$y=\sqrt{\frac{DD+zz}{4}}:+\frac{1}{2}z-\frac{1}{2}D$ Which is the *second Theorem.*

Q. E. D.

The Geometrical Effection of the *first* of theſe *Theorems* is very eaſie; for, by the 14th Step, 'tis evident that there are *three* Lines given to find a *fourth* proportional Line. [By *Problem* 3, *Page* 308.]

Scholium.

From the *Compariſons* which have been all-along made in this Chapter, between the *Hyperbola* and the *Ellipſis*, 'twill be eaſie *(even for a Learner)* to perceive the Coherence that is in (or between) thoſe two Figures; but, for the better underſtanding of what is meant by the *Centre* and *Aſymptotes* of an *Hyperbola*, conſider the annex'd *Scheme*, wherein it is evident *(even by Inſpection)* that the oppoſite *Hyperbola's* will always be alike, becauſe they will always have the ſame Tranſverſe Diameter common to both, *&c.* (ſee *Sect.* 1. of this *Chap.*) Alſo, that the middle Point, or common Centre of the *Ellipſis* is the common Centre to all the four *Conjugal Hyperbola's.*



And the Two Diagonals of the Right-angled *Parallelogram*, which circumſcribes the *Ellipſis*, *(or is inſcrib'd to the four* Hyperbola's) being continued, will be ſuch *Aſymptotes* to thoſe *Hyperbola's* as are defined *Chap.* 1, *Sect.* 5, *Defin.* 4.

Sect.

Sect. 6. *To draw the* Asymptotes *of an* Hyperbola, &c.

Having found the *Latus Rectum* (by Sect. 2.) and the Conjugate Diameter nSN in its true *Position*, by Sect. 3. Then, thro' the Centre C of the *Hyperbola*, and the Extream Points n N of its Conjugate Diameter, draw *Two Right-lines*, as CN and Cn, infinitely continued, *(as in the following Figure)* and they will be the *Asymptotes* requir'd.

That is, they are Two such Right-lines as, being infinitely extended, will continually incline to the Sides of the *Hyperbola*, but never touch them.

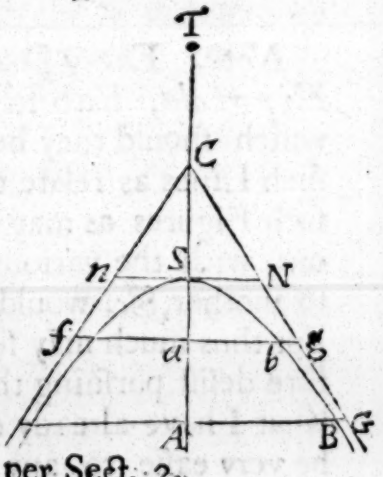
Demonstration.

Suppose the Semi-ordinates ab and AB to be *rightly apply'd* to the Axis TA, and produced both Ways to the *Asymptotes*, as at fg and FG; then will the $\triangle CSN$, $\triangle Cag$, and $\triangle CAG$ be alike.

Let $d = CS = TC$. And $L =$ the *Latus Rectum*; as before.

Put $\left\{\begin{matrix} e = Sa \\ y = SA \end{matrix}\right\}$ the *Abscissa's*. Then $\left\{\begin{matrix} d + e = Ca. \\ d + y = CA. \end{matrix}\right.$

Then	1	$d : SN :: d + e : ag$. viz. $CS : SN :: Ca : ag$
1 in □'s	2	$dd : \square SN :: dd + 2de + ee : \square ag$
But	3	$\frac{1}{2} dL = \square SN$. per Sect. 3.
2, 3 ∴	4	$\frac{ddL + 2deL + eeL}{2d} = \square ag$
Again	5	$2d : L :: 2de + ee : \square ab$. per Sect. 2.
5 ∴	6	$\frac{2deL + eeL}{2d} = \square ab$
4 − 6	7	$\frac{dL}{2} = \square ag - \square ab$
But	8	$ag + ab = bf$
	9	$ag - ab = bg$ } per *Fig.*
8 × 9	10	$\square ag - \square ab = bf \times bg$
7, 10	11	$bf \times bg = \frac{1}{2} dL$
Again	12	$dd : \square SN :: dd + 2dy + yy : \square AG$
That is,		$\square CS : \square SN :: \square CA : \square AG$
3, 12 ∴	13	$\frac{ddL + 2dyL + yyL}{2d} = \square AG$
But	14	$2d : L :: 2dy + yy : \square AB$. per Sect. 2.
14 ∴	15	$\frac{2dyL + yyL}{2d} = \square AB$



13 — 15	16	$\frac{dL}{2} = \square AG - \square AB$
Also	17	$AG + AB = BF$
	18	$AG - AB = BG$ per *Fig.*
17 × 18	19	$\square AG - \square AB = BF \times BG$
16, 19	20	$BF \times BG = \frac{1}{2} dL$
11, & 20 ∴	21	$bg = \frac{\frac{1}{2} dL}{bf}$. And $BG = \frac{\frac{1}{2} dL}{BF}$.

From the last Step 'tis evident, that the *Asymptotes* are nearer the *Hyperbola* at G than at g, and consequently will continually approach to its Curve: For BF) $\frac{1}{2} dL$ ($= BG$ is less than bf) $\frac{1}{2} dL$ ($= bg$, because the *Divisor* BF is greater than the *Divisor* bf; and it must needs be so, where-ever the *Ordinates* are produc'd to the *Asymptotes*, from the Nature of the *Triangles*.

Again; From the 7th and 16th Steps 'tis evident, that the *Asymptotes* can never really meet and be co-incident with the Curve of the *Hyperbola*, altho' both were infinitely extended, because $\frac{1}{2} dL$ will always be the Difference between the Square of any *Semi-ordinate* and the Square of that *Semi-ordinate*, when 'tis produc'd to the *Asymptote*.

Corollary.

From hence it follows, that every *Right-line* which passes thro' the *Centre*, and falls within the *Asymptotes*, will cut the *Hyperbola*; and all such Lines are call'd *Diameters*, (as in the *Ellipsis*) because the Properties of the *Hyperbola* and *Ellipsis* are the same.

Note. Every Diameter, both in the *Ellipsis*, *Parabola*, and *Hyperbola*, hath its particular *Latus Rectum* and *Ordinates*; which should they be distinctly handled, and the Effection of all such Lines as relate to them, as also the Nature and Properties of such Figures as may be *inscrib'd* and *circumscrib'd* to all the Sections, with the various Habitude or Proportions of one *Hyperbola* to another, *&c.* would afford Matter sufficient to fill a large Volume. But thus much may suffice by way of *Introduction*; I shall therefore desist pursuing them any farther, being fully satisfied, that if what I have already done be well understood, the rest must needs be very easie to any one that intends to proceed farther on that Subject.

AN

AN INTRODUCTION TO THE Mathematicks.

PART V.

THE Method of finding out any particular *Quantity* (viz. *either any* **Line**, **Superficies**, *or* **Solid**) by a Regular Progreſſion, or Series of *Quantities* continually approaching to it, which being *infinitely* continued, would then become perfectly *Equal* to it; is what is commonly call'd *Arithmetick*, which I ſhall briefly deliver in the following *Lemma's*, and apply them to Practice in finding the *ſuperficial* and *ſolid Contents* of *Geometrical Figures* farther on.

LEMMA I.

In any Series of Equal Numbers, (repreſenting Lines or other Quantities) *as,* 1. 1. 1. 1. &c. *Or* 2. 2. 2. 2. &c. *Or* 3. 3. 3. 3. &c. *if one of the Terms be multiply'd into the Number of Terms, the Product will be the Sum of all the Terms in the Series.*

This is ſo very plain, and eaſie to be underſtood, that it needs no *Example.*

LEMMA II.

If the Series *of Numbers in Arithmetick Progreſſion begin with a Cypher, and the common Difference be* 1; *as,* 0. 1. 2. 3. 4. &c. (repreſenting a Series of Lines or Roots beginning with a Point) *if the laſt Term be multiply'd into the Number of Terms, the Product will be double the Sum of all the* Series.

That is, putting $L =$ the *laſt Term*, $N =$ the *Number* of *Terms*, and $S =$ the *Sum* of all the *Series*:

Then

Then will $NL = 2S$. Conſequently, $\frac{1}{2}NL = S$. *viz.* one half of ſo many times the *greateſt Term* as there are Number of *Terms* in the Series.

Thus $\left\{\begin{matrix}0+1+2+3+4 \\ 4+4+4+4+4\end{matrix}\right. = \frac{10}{20} =$ the Sum of the Series $= \frac{1}{2}NL$
$= NL.$

And this will always be ſo, how many *Terms* ſoever there are, by *Conſect.* 1, *page* 185.

LEMMA III.

If a Series *of* Squares *whoſe Sides or Roots are, in* Arithmetick Progreſſion, *beginning with a Cypher,* &c. (as in the laſt Lemma) *be infinitely continued, the laſt Term being multiply'd into the Number of Terms, will be Triple to the ſum of all the* Series, *viz.* $NLL = 3S$, *or* $\frac{1}{3}NLL = S$.

That is, the Sum of ſuch a Series will be One Third of the *laſt* or *greateſt Term*, ſo many times repeated as is the number of *Terms* in the Series.

Inſtances in ſquare Numbers.

1. $\left\{\begin{matrix}0+1+4 \\ 4+4+4\end{matrix}\right. = \frac{5}{12} = \frac{1}{3} + \frac{1}{12}$.

2. $\left\{\begin{matrix}0+1+4+9 \\ 9+9+9+9\end{matrix}\right. = \frac{14}{36} = \frac{7}{18} = \frac{1}{3} + \frac{1}{18}$

3. $\left\{\begin{matrix}0+1+4+9+16 \\ 16+16+16+16+16\end{matrix}\right. = \frac{30}{80} = \frac{3}{8} = \frac{9}{24} = \frac{1}{3} + \frac{1}{24}$ &c.

From theſe Inſtances 'tis evident, that as the Number of *Terms* in the Series does encreaſe, the Fraction or Exceſs above $\frac{1}{3}$ does decreaſe, the ſaid Exceſs always being $\frac{1}{6N-6}$; which, if we ſuppoſe the Series to be *infinitely continued*, will then become *infinitely ſmall, viz.* in Effect nothing at all.

Conſequently, $\frac{1}{3}NLL$ may be taken for the true or perfect Sum of ſuch an *infinite* Series of Squares.

LEMMA IV.

If a Series of Cubes whoſe Roots are, in Arithmetick Progreſſion, *beginning with a Cypher,* &c. (as above) *be infinitely continu'd, the ſum of all the* Series *will be* $\frac{1}{4}NLLL = S$.

That is, One Fourth of the *laſt* or *greateſt Term* ſo many times repeated as is the Number of Terms.

Inſtances

Instances in Cube Numbers.

If 0 . 1 . 2 . 3 . *&c.* be the *Roots* of the *Cubes.*

Then 1. $\left\{\frac{0+1+8+27}{27+27+27+27} = \frac{36}{108} = \frac{4}{12} + \frac{1}{4} + \frac{1}{12}\right.$

2. $\left\{\frac{0+1+8+27+64}{64+64+64+64+64} = \frac{100}{320} = \frac{10}{32} = \frac{5}{16} + \frac{1}{4} + \frac{1}{16}\right.$

3. $\left\{\frac{0+1+8+27+64+125}{125+125+125+125+125+125} = \frac{225}{750} = \frac{45}{150} = \frac{3}{10} = \frac{6}{20} + \frac{1}{4} + \frac{1}{20}\right.$

From these *Examples* it plainly appears, that as the *Number* of *Terms* in the Series *encreases*, the *Fraction* or *Exceſs* above $\frac{1}{4}$ *decreaſes*, the *Exceſs* being always $\frac{1}{4N-4}$; which, if we ſuppoſe the Series to be *infinitely continu'd*, will become *infinitely ſmall*, or rather nothing: As in the laſt *Lemma.*

Conſequently, $\frac{1}{4}$ *NLLL* may be taken for the true and perfect Sum of all the *Terms* in ſuch an *infinite* Series of *Cubes.*

LEMMA V.

If a Series of Biquadrats, whoſe Roots are in Arithmetick Progreſſion, *beginning with a Cypher,* &c. (as before) *be infinitely continued, the Sum of all the Terms in ſuch a Series will be* $\frac{1}{5}NL^4$.

The Truth of this may be manifeſted by the like Proceſs as in the foregoing *Lemma's*, and ſo on for higher Powers. But if any one deſires a farther *Demonſtration* of theſe Series, he may (*I preſume*) meet with ample Satisfaction in Dr. *Wallis*'s Hiſt. of *Algebra*, chap. 78 & 79, wherein the Dr. concludes with theſe words:

" Thus having ſhew'd, that in a Progreſſion of Laterals (*or Arithmetical Proportionals*) beginning at 0. the ſum of 2. 3. 4. 5. 6 Terms, is always equal to half of ſo many times the greateſt; and there being no Pretence of Reaſon why we ſhould then doubt it in a Progreſſion of 7. 8. 9. 10. *&c.* we conclude it ſo to be, tho' ſuch Number of Terms be ſuppos'd infinite.

" Again; In a Progreſſion of their Squares having ſhew'd, that in 2. 3. 4. 5. 6 Terms the Aggregate is always more than One Third of ſo many times the greateſt, and the Exceſs always ſuch

" aliquot

" aliquot Part of the greateſt, as is denominated by ſix times the " Number of Terms wanting 1. (As, if the Terms be 2, " it is $\frac{1}{3} + \frac{1}{6}$; if 3, it is $\frac{1}{3} + \frac{1}{12}$; if 4, it is $\frac{1}{3} + \frac{1}{18}$; if 5, " it is $\frac{1}{3} + \frac{1}{24}$ of ſo many times the greateſt Term, and ſo on- " ward) we may well conclude, (there being no Pretence of " Reaſon why to doubt it in the reſt) that it will be ſo, how ma- " ny ſoever be ſuch Number of Terms. And becauſe ſuch Exceſs " as the Number of Terms do encreaſe will become infinitely " ſmall, (or leſs than any aſſignable) we conclude (from the " Method of Exhauſtions) that, if the Number of Terms be ſup- " poſ'd infinite, ſuch Exceſs muſt be ſuppoſ'd to vaniſh, and the " Aggregate of ſuch infinite Progreſſion ſuppoſ'd equal to $\frac{1}{3}$ " of ſo many times the greateſt.

" In like manner having prov'd that ſuch Progreſſion of Cubes " doth (as the number of Terms encreaſe) approach infinitely near " to $\frac{1}{4}$ of ſo many times the greateſt, and of Biquadrats to $\frac{1}{5}$, and " ſo of Surſolids to $\frac{1}{6}$ of ſo many times the greateſt, and ſo on- " wards as we pleaſe to try; and there being no Pretence of Rea- " ſon why to doubt it as to the reſt, we may take it as a ſufficient " Diſcovery, that (univerſally) the Aggregate of ſuch infinite " Progreſſion is equal (or doth approach infinitely near) to ſuch a " Part of ſo many times the greateſt, as is denominated by the " Exponent (or Number of Dimenſions) of ſuch Power (as is " that according to which the Progreſſion is made) encreas'd by " 1. namely, of Laterals $\frac{1}{2}$; of Squares $\frac{1}{3}$; of Cubes $\frac{1}{4}$; of Bi- " quadrats $\frac{1}{5}$; (of ſo many times the greateſt) and ſo onwards in- " finitely.

This Diſcourſe of the *Doctor*'s I thought convenient to inſert, ſuppoſing it may give ſome ſatisfaction to the Learner, to hear ſo Great a Man as Dr. *Wallis*'s Arguments about the Truth of theſe Series, which I have briefly deliver'd in the foregoing *Lemma*'s.

LEMMA VI.

If any Two Series *or* Ranks *of* Proportionals *have the ſame Num- ber of Terms*, (whether Finite *or* Infinite) *it will always be* { *As the firſt Term of one Series : is to the firſt Term of the other Series : : ſo is the Sum of all the Terms in the one Se- ries : to the Sum of all the Terms in the other Series.*

(12 e. 5)

As

As in these *Numbers*,		Or these *Numbers*,	
1	3	4	5
2	6	12	15
3	9	36	45
4	12	108	135
5	15	324	405
6	18	972	1215

That is, 1 : 3 : : 21 : 63 And 4 : 5 : : 1456 : 1820 &c.

The Application of these *Lemma's* to *Geometrical Quantities*, *viz.* to *Lines*, *Superficies*, and *Solids*, wholly depends upon granting the following *Hypotheses*.

The Hypotheses.

1. That every Line is suppos'd to consist (or be compos'd) of an *infinite Series* of Equidistant *Points*.

2. A *Surface* (viz. *the Area of any Figure*) to consist of an infinite Series of *Lines*, either *streight* or *crooked*, according as the *Figure* requires.

3. A *Solid* to consist of an infinite Series of *Plains*, or *Superficies*, according as its *Figure* requires.

Not that we suppose Lines, which have really no *Breadth*, can fill a Space or Superficies; or, that Plains, which have not any *Thickness*, can constitute a Solid: But by what we here call Lines are to be understood small *Parallelograms (or other Superficies) infinitely* narrow, yet so, as that their *Breadths* being all taken and put together, must be *Equal* to the *Figure* they are suppos'd to fill up.

And those *Plains* or *Superficies*, which are here said to constitute a *Solid*, are to be understood *infinitely thin*; yet so, as that their *depths* or *thicknesses* (*which are hereafter also call'd* Lines) being all taken together, must be equal to the *height* of the propos'd *Solid*.

Now, in order to render this *Hypothesis* as easie for a Learner to understand as I can, I shall here propose a very plain and familiar *Example*;

Viz. Let us suppose any *Book* to be compos'd (or made up) of 100, 200, 300 *(more or less)* Leaves of fine *Paper*; such a *Book* being close put together, will have *length*, *breadth*, and *depth* or *thickness*, and therefore may *(not improperly)* be call'd a *Solid*; and each of its Edges *(being evenly cut)* will be a *Superficies* compos'd of a *Series* of small *Parallelograms*, every one of their *breadths* being only the Edge of a single Leaf of Paper; and if we conceive the *Thickness* of every one of those Leaves to be

divided into 10, or 100, or 1000, &c. they will then become such a Series of *infinitely-small* Lines as are *(by the Hypothesis)* said to compose or fill up a *Superficies*.

And all the Superficies of those *infinitely-thin* or *divided* Leaves of *Paper*, will become such a Series of *Plains*, or *Superficies*, as are said to constitute a *Solid*, *viz.* such a *Solid* as the Bigness and Figure of that *Book*.

Now, according to this *Idea* of Lines, Superficies, and Solids, one may, without the least Prejudice to any *Demonstration*, admit of the following *Definitions* and *Theorems*.

Definitions.

I. The *Area*'s of *Squares*, and all other *Parallelograms*, are compos'd or fill'd up with an *infinite* Series of equal *Right-lines*.

II. The *Area* of every plain *Triangle* is compos'd of an *infinite* Series of *Right-lines* parallel to its *Base*, and equally decreasing until they terminate in a Point at the *Vertical Angle*.

III. The *Area* of a *Circle* may be compos'd either of an *infinite* Series of *concentrick* or *parallel Circles*, or of an *infinite* Series of *Chord-lines* parallel to its Diameter, or of an innumerable multitude of *Sectors*.

IV. The *Area* of an *Ellipsis* may be compos'd either of an *infinite* Series of *Ordinates rightly apply'd*, or of an *infinite* Series of *Right-lines* parallel to its *Transverse Diameter*.

V. The *Area*'s of the *Parabola* and *Hyperbola* are compos'd of an *infinite* Series of *Ordinates*; or may also be compos'd of *Right-lines* parallel to its Axis, &c.

VI. A *Prism* is a *solid Body* contain'd or included within several equal *Parallelograms*, having its *Bases* or *Ends* equal, and alike; and it's generally nam'd according to the Figure of its *Base*: That is,

VII. A *Cube (or Solid like a Dye)* is a *Prism* bounded or included within Six equal *square Plains*.

VIII. A *Parallelopipedon* is a *Prism* that hath its Sides bounded or included within four equal *Parallelograms* and two square *Bases* or *Ends*.

IX. A *Cylinder (or* Solid, *like a Rolling-stone in a Garden)* is only a *round Prism*, having its *Bases* or *Ends* a perfect Circle.

X. The

X. The *Solidity* of every *Prism* is compos'd of an *infinite ſeries* of equal Plains, parallel and alike to that of its Baſe.

XI. A *Pyramid* is a Solid bounded or included within ſeveral plain *Triangles* ſet upon any *Polygonous Baſe*, having their *vertical Angles* all meeting together in a Point, call'd the *Vertex*, and takes its Name from the Figure of its Baſe, *viz.* if it has a ſquare Baſe, 'tis call'd a *ſquare Pyramid*; if a triangle Baſe, 'tis call'd a *triangular Pyramid*, &c.

XII. A *Cone* is only a round Pyramid, which hath been already defined in *page* 355, &c.

XIII. The Solidity of every *Pyramid* is compos'd or conſtituted of an *infinite ſeries* of *Plains*, parallel and alike to that of its Baſe, equally decreaſing until they terminate in a Point at the *Vertex*.

XIV. A *Sphere* or *Globe* (*viz.* a *Ball*) is a *Solid* bounded or included within one Regular Superficies, being form'd or generated by the *Rotation* of a Semi-circle about its *Diameter*, (*call'd the Axis of a Sphere*) and its Solidity is compos'd or conſtituted of an *infinite ſeries* of *Concentrick Circles*, whoſe Diameters are the *Chords* of that *Circle* by which it was form'd.

XV. A *Spheroid* (*or Egg-like Figure*) is a Solid bounded with one Regular Superficies, form'd by the *Rotation* of a *Semi-ellipſis* about its *Tranſverſe Diameter*, (*call'd the Axis of the Spheroid*) and its Solidity is conſtituted of an *infinite ſeries* of *Concentrick Circles*, whoſe *Diameters* are the *Ordinates* of that *Ellipſis* by which it was form'd.

XVI. There is another ſort of Solid call'd an *Oblate Spheroid*, being form'd by the *Rotation* of an *Ellipſis* about its *Conjugate Diameter*, and is like a flat *Turnep*.

XVII. If a *Semi-parabola* be turn'd about its Axis, 'twill form a Solid call'd a *Parabola Conoid*, being compos'd or conſtituted of an *infinite ſeries* of *Circles*, whoſe *Diameters* are the *Ordinates* of a *Parabola*.

XVIII. If a *Parabola* be turn'd about its Baſe, or greateſt *Ordinate*, 'twill form a Solid call'd a *Pyramidoid*, but moſt commonly a *Parabolick Spindle*, which will be conſtituted of an *infinite ſeries* of *Circles*, whoſe Diameters are *Right-lines* parallel to the *Parabola*'s Axis.

XIX. If an *Hyperbola* be turn'd about its Axis, 'twill form a Solid call'd an *Hyperbolick Conoid*, being conſtituted of an *infinite ſeries* of Circles whoſe Diameters are the *Ordinates* of the *Hyperbola*.

XX. The *Curve Superficies* of all Circular Solids (viz. *Cylinders*, *Cones*, *Spheres*, &c.) are compos'd of an *infinite ſeries* of the *Peripheries* of thoſe Circles which conſtitute their Solidities.

Upon theſe *Definitions* are grounded all the following *Theorems*; and therefore, if they were diligently compar'd with their reſpective Figures, it muſt needs be of great help to the Learner, and would *render* all that follows *very eaſie*; wherein I ſhall begin with what hath been *already demonſtrated*, by way of introducing the reſt.

THEOREM I.

The Area *of every Right-angled* Parallelogram *is obtain'd by multiplying the Length into its Breadth.*

That is, $BD \times FB =$ the *Area* of the *Parallelogram* $BDFG$, by *Lemma* 1, compar'd with *Definition* 1.

Example.

Suppoſe $BD = 26$, and $FB = 9$
Then $26 \times 9 = 234$ the *Area*.
See *Prob.* 1, *pag.* 339.

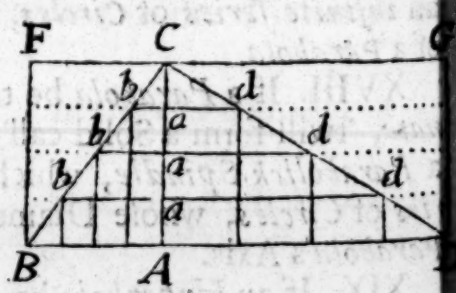
THEOREM II.

The Area *of every plain Triangle is equal to half the* Area *of its Circumſcribing* Parallelogram.

That is, $\frac{BD \times CA}{2} =$ *the* Area *of* $\triangle BCD$, *in the following Figure.*

Demonſtration.

Suppoſe the *Perpendicular* CA to be divided into an infinite number of equal *Parts*, as at the Points *a*, *a*, *a*, &c. and through thoſe Points there were drawn Right-lines parallel to the Baſe BD, (viz. $b\,a\,d$, $b\,a\,d$, $b\,a\,d$, &c.) Then will thoſe Lines be a *ſeries* of *Terms* in *Arithmetick Progreſſion*, beginning at the Point C, viz. 0, $b\,d$, $2\,b\,d$, $3\,b\,d$, &c. as is evident by the Figure, wherein BE is the *greateſt Term* $= L$, and CA the *Number* of *Terms* $= N$.



But

But $\frac{1}{2} NL = S$, by Lemma 2. And $S =$ the *Triangle*'s *Area* by Definition 2. Q. E. D.

Example. Let $BD = 26$, and $CA = 9$ As above,

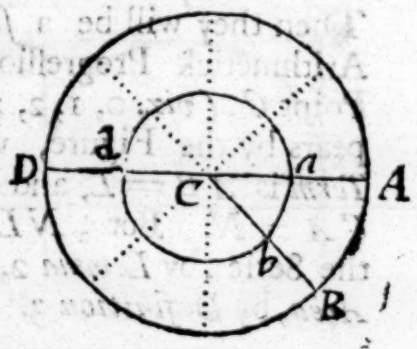
Then $\frac{26 \times 9}{2} = 117$. Or $\frac{26}{2} \times 9 = 117$. Or thus $26 \times \frac{9}{2} = 117$. the *Area* requir'd. [See *Problem* 3, *page* 330.]

THEOREM III.

The Peripheries of Circles are in Proportion one to another as their Diameters are.

Demonstration.

Let the *Periphery* of a *Circle* be divided into any Number of equal Arches by Right-lines drawn from the Centre, (viz. *Radius's*) suppose 'em 8, as in the annex'd Figure, wherein *AB* is one of them; Then, if thro' any Point in the Radius there be drawn a concentrick or parallel *Circle*, its *Periphery* will also be divided into 8 equal Arches by those *Radius's*, one whereof will be *a b*, and the $\triangle C a b$ will be like to $\triangle C A B$.



Therefore $Ca : ab :: CA : AB$. Or $Ca : CA :: ab : AB$

Consequently $2Ca : 2CA :: 8ab : 8AB$.

But $2Ca = da$ the Diameter of the Circle, whose *Periphery* is $8ab$.

And $2CA = DA$, the Diameter of the Circle, whose *Periphery* is $8AB$. Therefore, &c. as by *Theorem*. Q. E. D.

Example.

In Chapter 6, Part III, it was found, that if the *Diameter* of a Circle be 2, its *Periphery* will be 6,2831853, &c.

Ergo, 2 : 6,2831853, &c. :: 1 : 3,14159265, &c. the *Periphery* of the Circle whose *Diameter* is 1.

Corollary.

Hence it follows, that because *Unity*, or 1, may be made the *first Term* in the *Proportion*, therefore 3,14159265, &c. may be made a constant or settled *Factor*; which being multiply'd into any propos'd *Diameter*, will produce the *Periphery* of that Circle.

Note, *Instead of* 3,14159265, &c. *it may be sufficient to take only* 3,1416.

Or

Or, in whole Numbers the Proportion may be,

As 7 : 22 :. *Diam* :: *Periphery* } { *these Numbers may ſerve,*
Or 113 : 355 :: *Diam.* : *Periphery* } { *and are often uſed in common Practice.*

THEOREM IV.

The Area *of any* Sector *of a* Circle *is equal to half the Rectangle of the* Radius *into its Arch.*

That is, $\frac{CA \times AB}{2} =$ *the* Area *of* ACP.

Demonſtration.

Suppoſe the *Radius* CA to be divided into an *infinite ſeries* of *Equidiſtant Points*, as *a*, *e*, *y*, &c. and through thoſe Points there were drawn *concentrick* or *parallel Arches*, as *ab*, *ed*, *yf*, &c. Then they will be a *ſeries* of *Arches* in Arithmetick Progreſſion, beginning at the Point *C*, (*viz.* 0, 1, 2, 3, &c.) as plainly appears by the Figure, wherein the *greateſt Term* is $AB = L$, and *Number* of *Terms* is $CA = N$. But $\frac{1}{2} NL = S$ the Sum of all the Series, by *Lemma* 2, and $S =$ the Sector's *Area*, by *Definition* 3. Q. E. D.

Example.

Let the *Radius* $CA = 12$ And the Arch $AB = 8$.

Then $\frac{12 \times 8}{2} = 48$. Or $\frac{12}{2} \times 8 = 48$. Or $\frac{8}{2} \times 12 = 48$, the *Area* of the Sector ACB.

THEOREM V.

The Area *of every Circle is equal to half the Rectangle of the* Radius *into its* Periphery.

That is, according to Archimedes, *a Circle is equal to a Right-angled Triangle, whoſe Sides containing the Right-angle are equal, one to the* Radius, *and the other to the* Perimeter *of that Circle.* Pro 1. de Dimenſione Circuli.

The Truth of this *Theorem* may be eaſily deduced from the *laſt* thus; If we ſuppoſe the laſt Sector to be one Eighth part of a Circle, then it follows, that $\frac{8 AB \times CA}{2} = 4 AB \times CA$ will be the *Area* of the whole Circle.

But $4 AB =$ half the Circle's Periphery, and $CA =$ half its Diameter; Therefore, &c. As *per* Theorem. Q. E. D.

Exam-

Example.

If the *Diameter* be *Unity*, or 1, the *Periphery* will be 3,14159265 *&c.* by *Theorem* 3.

Then $\frac{3,14159265}{2} \times \frac{1}{2} = 0,78539816$, &c. (*or* 0,7854 *for common Use*) will be the *Area* of that Circle.

Scholium

From hence naturally flows the following *Proportion* between the Square and its inscrib'd Circle.

Proportion. { *As the* Perimeter (*viz.* the Sum of the Four Sides) *of any Square* : *is to its* Area :: *so is the* Periphery *of the Inscrib'd Circle to its* Area.

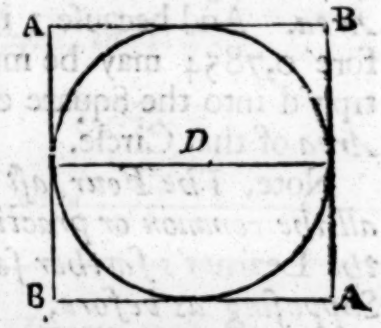
That is, supposing $A\ B = D =$ the Side of the Square, and the Diameter of its *Inscrib'd Circle*;

Then $4D =$ the Perimeter, $DD =$ the *Area* of the Square, and $3,1416D =$ the *Periphery* of the Circle. By *Theorem* 3.

But $4D : DD :: 3,1416D : 0,7854DD =$ the Circle's *Area*.

And if $D = 1$; Then $4D = 4$ and $DD = 1 \times 1 = 1$; and the *Periphery* will be 3,1416

Then 4 : 1 :: 1 : 0,7854 &c. As in the *Example* above.



And from hence may be easily deduced the following *Theorem.*

THEOREM VI.

The Area's *of all Circles are in proportion one to another as the Squares of their Diameters.* (2. *e.* 12)

For if $D =$ the Diameter of one Circle, and $d =$ the Diameter of another Circle,

Then will 0,7854 DD be the *Area* of one Circle, and 0,7854 dd will be the *Area* of the other Circle; as above.

But 0,7854 DD : 0,7854 dd : : DD : dd. Or thus,

Let $D =$ the Diameter, and $P =$ the Periphery of one Circle; $d =$ the Diameter, and $p =$ the Periphery of another Circle;

Then	1	$\frac{1}{2}D \times \frac{1}{2}P = \frac{1}{4}DP = A$ *the* Area *of one Circle.*
And	2	$\frac{1}{2}d \times \frac{1}{2}p = \frac{1}{4}dp = a$ *the* Area *of the other Circle*
1 × 4	3	$DP = 4A$ (*per last Theorem*
2 × 4	4	$dp = 4a$
3 ÷ D	5	$P = \frac{4A}{D}$

4 ÷ d

$4 \div d$	6	$p = \frac{4a}{d}$
But	7	$P : p :: D : d$. *Per Theorem* 3.
5, 6, 7	8	$D : d :: \frac{4A}{D} : \frac{4a}{d}$
8 ∵	9	$4DDa = 4ddA$. That is, $DDa = ddA$
9, *Analogy*	10	$DD : A :: dd : a$. Or $A : a :: DD : dd$

Q. E. D.

Corollary.

Hence it follows, that becauſe the Square of 1 is 1, (*viz.* $1 \times 1 = 1$) and 0,78539816, &c. or 0,7854 is the *Area* of the Circle whoſe Diameter is 1, (*as before*) therefore it will be 1 : 0,7854 :: So is the Square of any Circle's Diameter : to its *Area*. And becauſe 1 is the *firſt Term* in the *Proportion*, therefore 0,7854 may be made a conſtant *Factor*; which being multiply'd into the Square of any propos'd Diameter, will produce the *Area* of that Circle.

Note, *The Four laſt Theorems do plainly ſhew the Reaſon of all the common or practical* Problems *about a* Circle, *which, for the* Learner's *farther ſatisfaction, I have here inſerted together. Suppoſing as before,*

That $\left\{\begin{array}{l} D = \text{the } Diameter \\ P = \text{the } Periphery \\ A = \text{the } Area \end{array}\right\}$ of any propos'd Circle;

Then

Probl. 1. D *being given, To find* P.

	1	$1 : 3,1416 :: D : P$ *per* Theorem 3.
1 ∵	2	$3,1416\ D = P$
Examp.		Suppoſe $D = 32$. Then $3,1416 \times 32 = 100,5312$ the *Periphery*.

Probl. 2. D *being given, To find* A.

	3	$1 : 0,7854 :: DD : A$ *per* Theorem 6.
3 ∵	4	$0,7854\ DD = A$
Examp.		Suppoſe $D = 32$, (as before)
Then		$DD = 32 \times 32 = 1024$
And		$0,7854 \times 1024 = 804,2496$ the *Area* requir'd.

Probl. 3. P *being given, To find* D.

$2 \div$	5	$D = \frac{P}{3,1416}$ Or $\left\{\begin{array}{l} \text{becauſe } \frac{1}{3,1416} = 0,3183 \\ \text{therefore } 0,3183\ P = D. \end{array}\right.$

This being only *Converſe* to the *firſt*, needs no *Example*.

1 G

Prob. 4. *P, being given, To find A.*

2 ⊙ 2	6	$9,86965\ DD = PP$
6 ÷	7	$DD = \frac{PP}{9,86965}$ Or $0,10132\ PP = DD$
4 ÷	8	$DD = \frac{A}{0,7854}$ Or $1,2732\ A = DD$
For		$\frac{1}{0,7854} = 1,2732$
7, 8	9	$\frac{PP}{9,86965} = \frac{A}{0,7854}$ Or $0,10132\ PP = 1,2732\ A$
9 × &c.	10	$\frac{PP}{12,5664} = A$ Or $0,07957\ PP = A$

Prob. 5. *A being given, To find D.*

8 ww 2	11	$D = \sqrt{\frac{A}{0,7854}}$ Or $D = \sqrt{1,2732\ A}$

Prob. 6. *A being given, to find P.*

10 × &c.	12	$PP = 12,5664A$ Or $PP = \frac{A}{0,07657}$
12 ww 2	13	$P = \sqrt{12,5664A}$ Or $P = \sqrt{\frac{A}{0,07957}}$

These *Six Problems* contain all the *Variety* that can be propoſed about finding the *Periphery*, *Diameter*, and *Area* of any *Circle*.

But if it be requir'd to find the *Area* of any *Segment*, or Part of a *Circle* cut off by a *Chord*, that Work will require a farther Conſideration.

Firſt, As to the *Data*, there muſt always be given the *Diameter*; or, either the *Periphery* or *Area* of the *Circle*, in order to find the *Diameter*.

Secondly, There muſt alſo be given, either the *Chord*, which is the *Baſe* of the *Segment*, or the *verſed Sine*, which is the *Height* of the *Segment*.

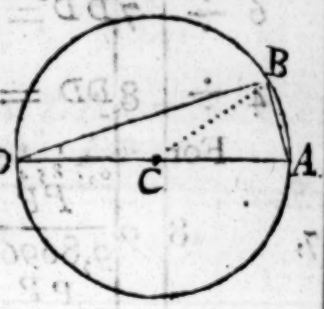
That is, either *B G*, or *A F*, in the following *Scheme*, muſt be given, that ſo the *Area* of the △ *B C G* may be found.

Then it's evident, *(by the Figure)* that if the *Area* of the △ *B C G* be taken from the *Area* of the *Sector C B A G*, the *Remainder* will be the *Area* of the *Segment B A G*.

And if the *Area* of the *Segment B A G* be taken from the whole *Area* of the *Circle*, the *Remainder* will be the *Area* of the other *Segment D B C*.

Example in *Numbers.*

Let there be given $DA = 32$, as in *Prob.* 1
And the *verſed Sine* $AF = 6$.
Then $\frac{1}{2}DA = BC = CA = 16$.
And $CA - AF = CF = 10$.
But $\square BC - \square CF = \square BF$.
Conſequently $\sqrt{\square BC - \square CF} = BF$
Viz. $\sqrt{156} = 12{,}49 = BF$.



Then by the Doctrine of plain *Triangles*, the *Arch* $BA = \angle BCA$ may be found in *Degrees* and *Decimal Parts*.
Thus BC : *Radius* :: BF : *Sine* $\angle BCF = 51°, 31$ *Degrees*.
And then it will always hold in this *Proportion*;

Viz. { *As the Circle's Periphery in Degrees* : *is to its Periphery in Equal Parts* (according to the Dimenſions taken) : : *So is the Arch in Degrees* (viz. $\angle BCA$) : *To the ſame Arch in Equal Parts.*

That is, $360° : 100{,}5312 :: 51°{,}31 : 14{,}3284 = BA$.
Then $14{,}3284 \times 16 = 229{,}2544$ the *Area* of the *Sector* $BCAG$.
And $12{,}49 \times 10 = 124{,}9$ the *Area* of the $\triangle BCG$.
Their Difference $104{,}3544$ = the *Area* of the *Segm.* BAG.
Or the *Area* of any *Segment* may be otherwiſe found (*as moſt uſually it is*) by a *Table* of the *Segments* of a *Circle*, whoſe *Area* is *Unity*, or 1. The *Conſtruction* or making of ſuch a *Table* is very well laid down in Mr. *Darie's Book* of *Gauging*, *Chap.* 9. which he performs in this *Problem.*

PROBLEM.

In a Circle whoſe Area is Unity, and its Diameter cut by Chord Lines into 1000 *Equal Parts, To find the Segment to any verſed Sine propos'd, not exceeding* 500 *of thoſe Equal Parts.*

1. Multiply the *verſed Sine* propos'd by 0,002, and *Subſtract* the *Product* from an *Unit* or 1.

2. This *Remainder* you ſhall ſeek in the *Common Table* of *Natural Sines*, (*the Arch being Divided into Degrees and Centeſimals*) which being found, let its *Co-arch* be doubled, and call'd *A*.

3. You muſt find the *correſpondent Sine* to *A*; which *Sine* being found, you may call *S*, and then it holds
$6{,}2831853) \; 0{,}0174532925\, A - S$ (= the *Segment* requir'd.

Now

Now this *Segment* being thus found, if you *Subduct* it from an *Unit*, you have the *Co-segment*, &c.

Note, *Notwithstanding what has been said in the second* Precept *of this* Problem, *it very often falls out that the Remainder there spoken of cannot be truly found in the Table of Natural Sines; therefore in this Case my Advice is, that you make Two Operations, one with a Sine the next Greater, and one with a Sine the next Less; and in so doing you will be sure to have the Segment requir'd bounded between the Results of those Two Operations.*

Example, *Let it be propos'd to find the Correspondent Segment to the versed Sine* 263.

First, 263 × 0,002 = 0,526, and 1 − 0,526 = 0,474, its *Arch is* 28°,29 being *Less* than just; its *Complement* is 61,71 which being doubled, is 123,42 = *A*.

Then ,0174533 *A* = 2,154086286
− 0,8346556 = *S* The *Sine* of *A*.

6,2831853) 1,319430686 (0,209993 the *Segment*.

Now I make a second Work.

263 being *Multiply'd* with 0,002 is 526. and 1 − 526 = 0,474 its *Arch* is 28°,30 being greater than just; and its *Complement* is 61,70; which being doubled is 123,4 = *A*.

Then 0,0174533 *A* = 2,1537372
− 0,8348478 = *S* the *Sine* of *A*

6,2831853) 1,3188894 (0,29907 the *Segment*.

So you see by these *Two Operations*, that the *Segment* is bounded, and 'tis very probable it may be 0,20995.

But to abbreviate this *Large Factor*, and this *Large Divisor*, I shall here insert Two *Tablets* of them, which will be ready for Use, and *Exact* enough too.

Divisor.		*Factor.*	
6,2832	1	,0174533	1
12,5664	2	,0349066	2
19,8495	3	,0523599	3
25,1327	4	,0698132	4
31,4159	5	,0872665	5
37,6991	6	,1047197	6
43,9823	7	,1221730	7
50,2655	8	,1396263	8
56,5487	9	,1570796	9

Thus far, Mr. *Darie*, which I have here inserted to shew the *Learner* how, by the Help of these two *Tablets*, and a *Table* of *Natural Sines*, he may easily make a *Table* of *Segments*, whose Use shall be shew'd farther on, *viz.* when I come to treat of *Practical Gauging*. In the mean Time I shall here lay down another Method

To find the *Area* of any *Segment* of a *Circle* (*very near*) by a *New Theorem*, without the Help either of a *Table* of *Sines* or *Segments*, having the same *Data* as before in *Page* 404.

Viz. Let $\begin{cases} R = \text{the } \textit{Radius}, \text{ or } \frac{1}{2} \textit{Diameter} \text{ of the given } \textit{Circle}. \\ d = \text{the } \textit{Difference} \text{ between the } \textit{verſed Sine} \text{ and } \textit{Radius}. \\ C = \textit{half} \text{ the } \textit{Chord} \text{ of the } \textit{Segment's Baſe}. \end{cases}$

Theorem. $\left\{ \dfrac{2\frac{1}{3}RR - 1\frac{1}{3}Rd - dd}{1\frac{1}{2}R + d} \times = CS, \text{the } \textit{Area of the Segm.} \right.$

Example, Suppoſe $R = BC = 16$. $d = FC = 10$. and $C = BF = 12{,}49$. As before

Then $2\frac{1}{3}RR = 597{,}3333$ $1\frac{1}{3}Rd = 213{,}3333$. $dd = 100$

$- 313{,}3333 = 1\frac{1}{3}Rd + dd$

$1\frac{1}{2}R + d = 34)\ 284{,}0000\ (8{,}3529$

Laſtly, $8{,}3529 \times 1249 = 104{,}3276$ the *Area* of the *Segment B A G*, As before.

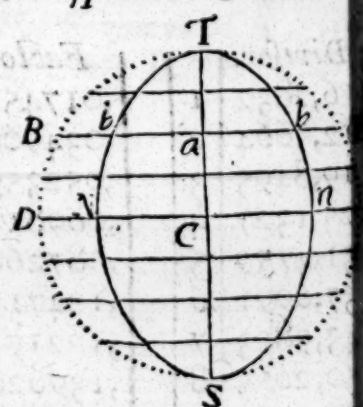
THEOREM VII.

As Squares are to the Area's *of their Inſcrib'd Circles, So are Parallelograms to the* Area's *of their Inſcrib'd Ellipſis.*

That is, $\begin{cases} \textit{As the Square of the Diameter of any Circle: is to its} \\ \text{Area} :: \textit{So is the Rectangle of the Tranſverſe and} \\ \textit{conjugate Diameters of any Ellipſis: to its} \text{ Area}. \end{cases}$

Demonſtration.

Circumſcribe any *Ellipſis* with a *Circle*; and ſuppoſe an *Infinite Number* of *Chord Lines* drawn therein, all parallel to the *Conjugate Diameter*; as thoſe in the annex'd *Figure*; then it will be $\begin{cases} \textit{As } (D\,A) \textit{ the Diameter of the Circle: Is to } (N\,n) \textit{ the} \\ \textit{Conjugate Diameter of the Ellipſis:: So is } (B\,a\,B) \textit{ any} \\ \textit{Chord in the Circle: To } (b\,a\,b) \textit{ its reſpective Ordinate} \\ \textit{in the Ellipſis.} \end{cases}$



For according to the *Property* of the *Circle*.

it is	1	$TS - Ta \times Ta = \square\, Ba$
And		by the *Property* of the *Ellipſis*
it is		$\square T : \square NC :: TS - Ta \times Ta : \square ba$
1, 2		$\square TC : \square NC :: TS - BA : \square ba$
3, Hence		$TC : NC :: Ba : ba$
Conſeq.		$2\,TC : 2\,NC :: 2\,Ba : 2ba$
That is,		$D\,A : Nn :: B\,a\,B : b\,a\,b$
Put		$D = 2TC$, and $d = 2NC$
Then	8	$D : d ::$ *Chord* $B\,a\,b$: *Ordinate* $b\,a\,b$, &c.

But

But the *Sum* of an *Infinite Series* of such *Chords*, as *B a B*, do constitute the *Area* of the *Circle*, by *Definition* 3.

And the *Sum* of the like *Series* of their respective *Ordinates*, as *b a b*, do constitute the *Ellipsis's Area*, by *Definition* 4.

Therefore *D* : *d* :: *Circle's Area* : *Ellipsis Area*, by *Lemma* 6.

But *D* : *d* :: *D D* : *D d*. Whence it follows,

That *D D* : *Circle's Area* :: *D d* : *Ellipsis's Area*, Q. E. D.

Consequently, As 1 : is to 0,7854 :: so is the *Rectangle*, or *Product* of the *Transverse*, and *Conjugate Diameters* of *any Ellipsis* : *To* its *Area*.

Example, Suppose *TS* = 36. and *Nn* = 16. Then 36 × 16 = 576. And 576 × 0,7854 = 452,3904 the *Area* of the *Ellipsis*.

Corollaries.

1. Hence it is easy to conceive, that the *Square Root* of the *Rectangle* or *Product* of the *Transverse*, and *Conjugate Diameters*, will be the *Diameter* of a *Circle* whose *Area* will be Equal to the *Ellipsis Area*.

Viz. √ 576 = 24 the *Diameter* of a *Circle* = to the *Ellipsis*.

2. All *Segments* of an *Ellipsis* and its *Circumscribing-Circle*, (*whose Bases are parallel to the Conjugate Diameter, and of the same Height*) are in *Proportion* one to another, as their *Bases* are. That is, *B a B* : *b a b* :: *Area Segment BNB* : *Area Segment bNb*; Or *TS* : *Nn* :: *Area Segment BNB* : *Area Segment b Nb*.

THEOREM. VIII.

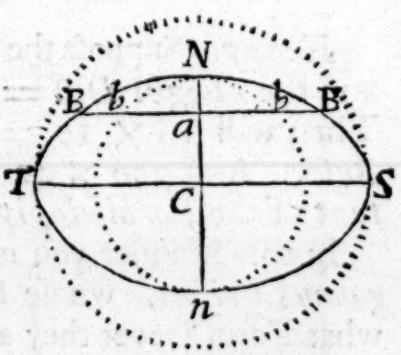
The Area *of every* Ellipsis, *is a mean Proportional between the* Area's *of its* Circumscribing, *and* Inscrib'd Circles.

The Truth of this *Theorem* may be easily deduc'd from the Last; for supposing *D* = *TS*, and *d* = *Nn*, as before. Then it is already prov'd, that *DD* : *D d* :: *Circumscribing Circle's Area* : *Ellipsis Area*.

But *DD* : *D d* :: *D d* : *d d*.

Therefore *Ellipsis Area* : *Inscrib'd Circle's Area* :: *D d* : *dd*.

By *Theorem* 6.



Example, Let *TS* = *D* = 36. and *Nn* = *d* = 16. As before. Then *DD* = 1296. and *dd* = 256.

Then

Then will $\begin{cases} 1296 \times 0{,}7854 = 1017{,}8784 \text{ the } \textit{great Circle's Area} \\ 256 \times 0{,}7854 = 201{,}0624. \end{cases}$

Suppose A = the *Ellipsis Area*. Then, according to the *Theorem*, it will be, $1017{,}8784 : A :: A : 201{,}0624$.

Ergo $AA = 1017{,}8784 \times 201{,}0624 = 204657{,}07401216$.

Consequently, $\surd\ 204657{,}07401216 = 452{,}3904 = A$, the *Area* of the *Ellipsis*, As before in the *Last Example*.

Corollary.

From hence it follows, that all *Segments* of an *Ellipsis* and its *Inscrib'd Circle*, *whose Bases are parallel to the Transverse Diameter*, *and have the same Height*, are in *Proportion* one to another as the *Area's* of the *Ellipsis* and *Circle* are.

That is, *Area* of *Circle* : *Area* of *Ellipsis* : : *Segment b N b* : *Segment B N B*.

Or, *N n* : *T S* : : *Area Segment b N b* : *Area Segment B N B*.

THEOREM IX.

The Solid Content of any Prism (what Figure soever its Base is of) *is obtain'd by multiplying the* Area *of its* Base *into its Height*.

For Instance, a *Parallelopipedon* (*or Square Prism*) is constituted of an *Infinite Series* of Equal *Squares*; that of its *Base B A* being one of the *Terms*, and its *Height D B*, or *G A*, the *Number* of all the *Terms*.

Consequently, the *Area* of $B A b a \times D B$ = the *Sum* all the *Series* (by *Lemma* 1.) which is the *Solidity* of the *Parallelopipedon D B G A*, by *Definition* 10.

Example, Suppose the Side of the *Base B A* = 16, and the *Height D B* = 42.
Then will $16 \times 16 = 256$ be the *Area* of the *Base*. And $256 \times 42 = 10752$ the *Solid Content* of the *Parallelopipedon D B G A*.

In this Manner you may find the *Solidity* of all regular *Polygonous Prisms*, whose *Bases* (*or Ends*) are *parallel* and alike, what Form soever they are of.

That is, whether their *Bases* are *Triangles*, *Pentagons*, *Hexagons*, or *Octagons*, &c.

THEO-

THEOREM X.

Every Pyramid is the Third Part of the Priſm, that hath the ſame Baſe and Height with it. (*7. e.* 12.)

That is, the *Solid Content* of the *Pyramid B V A*, (*in the laſt Figure*) is one *Third* of *its circumſcribing Priſm D B G A.*

Demonſtration.

For every *Pyramid* that hath a *ſquare Baſe* (*as B A b a, in the laſt Figure*) is conſtituted of an *Infinite Series of Squares*, whoſe *Sides* or *Roots* are continually *encreaſing* in *Arithmetick Progreſſion*, beginning at the *Vertex* or *Point V* (See Theor. 2) its *Baſe B A b a*, being the *Greateſt Term*, (= *LL*) and its *perpendicular Height V C*,

or *D B*, is the *Number* of all the *Terms* (= *N*); but $\frac{NLL}{3} = S$ the *Sum* of all the *Series*, by *Lemma* 3. and *S* = the *Solid Content* of the *Pyramid B V A*, By *Definition* 13.

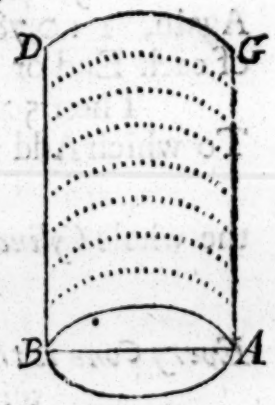
Example, Suppoſe the *Side* of a *Pyramid's Baſe be B A* = 16. and its *Height be V C* = 42. Then 16 × 16 = 256 the *Area* and its *Baſe B A b a*. And $\frac{256 \times 42}{3} = 3584$. Or $\frac{256}{3} \times 42 = 3584$ Or thus, $256 \times 4\frac{2}{3} = 3584$, is the *Solidity* of that *Pyramid B V A.*

Corollary.

From hence it will be Eaſy to conceive, that every *Pyramid* is $\frac{1}{3}$ of its *Circumſcribing Priſm*, what Form ſoever its *Baſe* is of, *viz.* whether it be a *Square*, *Triangle*, *Pentagon*, &c.

THEOREM XI.

The Solid Content of every Cylinder, is obtain'd by Multiplying the Area *of its Baſe into its Height.*



For every *Right Cylinder* is only a round *Priſm*, being conſtituted of an *Infinite Series* of equal *Circles*; that of its *Baſe* or End being one of the *Terms*, and its *Height B D* is the *Number* of all the *Terms.* Therefore the *Area* of its *Baſe B A* being *multiply'd* into *D B*, will be its *Solidity*, by *Lemma* 1. *Viz.* Let *D* = *B A*, and *H* = *G A*
Then 0,7854 *DD* × *H* = its *Solidity.*

Example,

Example, Let the *Diameter* of its *Base* be $D = 16$, and its *Height* $H = 42$.

Then 1 : 0,7854 : : $16 \times 16 = 256$: 201,0624 the *Area* of its *Base*.

And $201,0624 \times 42 = 8444,6208$ the *Solid Content* of that *Cylinder DBGA*.

Corollary.

Hence it is evident, that every *Square Parallelopipedon* is to its *Inscrib'd Cylinder*, As 1 : is to 0,7854. Or in whole *Numbers*, as 452 : to 355 very near.

And that all *Prisms* are in *Proportion* to their *Inscrib'd Cylinders*, as the *Area's* of their *Bases* are.

THEOREM XII.

The Curve Superficies of every Right Cylinder is Equal to the Rectangle made of its Height into the Periphery of its Base.

That is, *DB multiply'd* into the *Periphery* of the *Diameter BA*, will produce the *Curve Superficies* of the Last *Cylinder DG BA*.

For the *Cylinder* is *constituted* of an *Infinite Series* of Equal *Circles (according to the last Theor.)* therefore its *Curve Superficies* is compos'd of the *Peripheries* of those *Circles*, by *Definition* 20. But the *Periphery* of its *Base BA* is one of the *Terms*, and its *Height DB* is the *Number* of *Terms*. Therefore, &c. As by *Lemma* 1.

To which, if there be Added the *Area's* of both its Ends, *(or Bases)* the *Sum* will be the *Superficies* of the whole *Cylinder*.

Example. Suppose the *Diameter* of its *Base* to be $BA = 16$, and its *Height* $DB = 42$, As before;

Then 1 : 3,1416 : : 16 : 50,2656 the *Periphery* of its *Base*.

Again, 1 : 0,7854 : : $16 \times 16 = 256$: 201,0624 the *Area* of each End or *Base*.

Then $50,2656 \times 42 = 2111,1552$ the *Curve Superficies*.

To which Add $201,0264 \times 2 = 402,1248$ both the end *Area's*.

The *Sum* = 2513,2800 is the *Superficies* of the whole *Cylinder*.

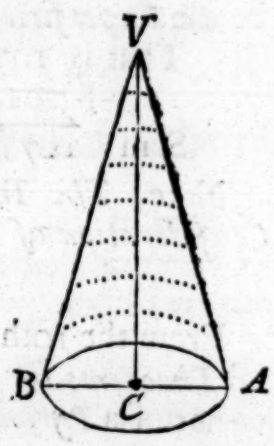
THEOREM XIII.

Every Cone is the Third Part of a Cylinder, having the same Base with it, and their Altitudes Equal. (10. *e.* 12)

Demon

Demonstration.

The Truth of this *Theorem* may be easily conceiv'd by only considering, that a *Cone* is but a *round Pyramid*, and therefore it must needs have the same *Ratio* to its *circumscribing Cylinder* as the *square Pyramid* hath to its *circumscribing Parallelopipedon*, *viz.* as 1 : to 3. However, to make it yet clearer, let it be farther consider'd, that



Every *Right Cone* is constituted of an *infinite Series* of *Circles*, whose Diameters do continually encrease in *Arithmetick Progression* beginning at the *Vertex* or *Point V*, the *Area* of its *Base BA* being the greatest *Term*, and its perpendicular Height *VC* the *Number* of all the *Terms*; therefore the *Area* of the Circle $BA \times \frac{1}{3} VC$ will be the Sum of all the *Series*, by *Lemma* 3, which is the *Cone*'s *Solidity*.

Example. Let the Diameter of its *Base* be $BA = 16$, and its Height $VC = 42$;
Then $1 : 0{,}7854 :: 16 \times 16 = 256 : 201{,}0624$ the *Area* of the *Base*. And $\frac{201{,}0624 \times 42}{3} = 2814{,}8736$ the *Solidity* of the *Cone BVA*. Or thus, $201{,}0624 \times \frac{42}{3} = 2814{,}8736$, &c.

Corollary.

Hence it follows that every square Pyramid is to its *inscrib'd* Cone as 1 : 0,7854. (Or as 452 : 355) Consequently, that all Pyramids have the same *Ratio* to their *inscrib'd Cones* as the *Area*'s of their Bases have.

THEOREM XIV.

The Curve Superficies of every Right Cone is equal to half the Rectangle of the Periphery of its Base into the length of its Side.

The Truth of this *Theorem* is self-evident from the *Definition* of a Cone, *Chap.* 1, *Part* IV, where it appears that the *Curve Superficies* of every Right Cone (*as BVA*) is equal to the *Area* of a *Sector* of that Circle whose *Radius* is the Side of the Cone (*VB*) and its *Arch* equal to the *Periphery* of the Cone's Base (*BA*). But the *Area* of any *Sector* is equal to half the *Rectangle* of the *Radius* into its *Arch*, by *Theorem* 4. Therefore, &c.

Example. Suppoſe the Length of the Cone's Side to be VB, or $VA = 42{,}7551$.

And the Diameter of its Baſe, *viz.* $BA = 16$, (As before)

Then will 50,2656 be the *Periphery* of its Baſe,

And $\frac{50{,}2656 \times 42{,}7551}{2} = 1074{,}5553$, &c. the *Curve* of the *Superficies*.

To which if there be added the *Area* of its Baſe, the Sum will be the *Superficies* of the whole (viz. *all the*) Cone.

That is 1074,5553
$+$ 201,0624 the *Area* of the Baſe.
Sum 1275,0177 is the total *Superficies*, &c.

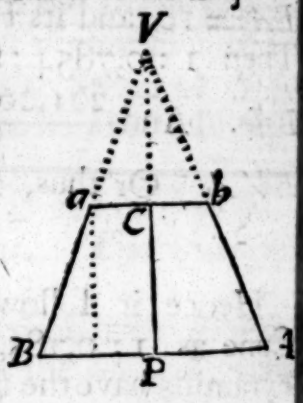
Note, *The Truth of this Theorem may be prov'd from the Conſideration of the laſt Theorem, and Definition* 20.

Scholium.

From the 10th and 13th *Theorems* may be eaſily deduced ſeveral *Theorems* for finding the *ſolid Content* of any *Fruſtum* or *Part* either of a *Pyramid* or *Cone*, cut by a Plain parallel to its *Baſe*.

Suppoſe a ſquare *Pyramid*, as BVA, to be cut by a *Plain* at ab, parallel to its *Baſe* BA, and it were requir'd to find the *Solidity* of the *Fruſtum* or *Part* $abAB$; Let there be given

$D = BA$ the Side of the Greater Baſe.
$d = ba$ the Side of the Leſſer Baſe.
$H = CP$ the perpendicular Height.



Firſt,	1	$D - d : H :: d : \frac{dH}{D-d} = VC$ by the Figure.
Then	2	$\left\{ DD \times \frac{H+VC}{3} = \text{the whole Pyramid } BVA. \right.$ By *Theorem* 10.
And	3	$dd \times \frac{1}{3} VC =$ the Pyramid aVb cut off.
Viz. 1, 2	4	$\left\{ \frac{DDDH}{3D-3d} = \text{the whole Pyramid } BVA. \right.$
And 1, 3	5	$\left\{ \frac{dddH}{3D-3d} = \text{the Pyramid } aVb. \right.$
4 — 5	6	$\left\{ \frac{DDDH - dddH}{3D-3d} = \text{the Fruſtum } abAB. \right.$
6, *Reduc.*	7	$DD + Dd + dd : \times \frac{1}{3} H =$ the Fruſtum $abAB$

Which in Words gives this following *Theorem*.

THEO

THEOREM XV.

To the Rectangle of the Sides of the two Baſes, *add the Sum of their* Squares; *that Sum being multiply'd into One Third of the* Fruſtum's *Height, will give its Solidity.*

Example. Suppoſe the Side of the greater Baſe $BA = 16$
And the Side of the leſſer Baſe (or Top) $ab = 12$
The Height $CP = 9$.
Then $16 \times 12 = 192$. $16 \times 16 = 256$. and $12 \times 12 = 144$.
Next $192 + 256 + 144 = 592$. and $\frac{592 \times 9}{3} = 1776$
Or $592 \times \frac{3}{1} = 1776$ the Content of the *Fruſtum* of a *Square Pyramid*.

And if it were the like *Fruſtum* of a *Right Cone*, it may be found by the ſame *Theorem*. Suppoſing D = the Diameter of the greater Baſe, d = the Diameter of the leſſer, and H = the Height of the *Fruſtum*;

Then being the Sum of all the *Squares* which conſtitute the *Fruſtum* of a *Square Pyramid*, are to the Sum of all the Circles which conſtitute the like *Fruſtum* of a right Cone, in the *Ratio* of 1 : to 0,7854 (or of 452 : to 355) Therefore it will be $1 : 0{,}7854 :: DD + Dd + dd \times \frac{1}{3}H : 0{,}7854\,DD + 0{,}7854Dd + 0{,}7854dd \times \frac{1}{3}H$ = the Cone's *Fruſtum*.

That is, in the laſt *Example*, $1 : 0{,}7854 :: 1776 : 1394{,}8704$ the like *Fruſtum* of a right Cone.

Or, becauſe $\frac{1}{0{,}7854} = 1{,}273236$, &c. Therefore it may be made $1{,}273236)\ DD + Dd + dd \times \frac{1}{3}H$ (= the ſame *Fruſtum*; That is, $1{,}273236)\ 1776\ (1394{,}87$, &c. As before.

And if you take the *Triple* of this Diviſor, *viz.* $1{,}273236 \times 3$ it will be $3{,}8197)\ DD + Dd + dd : \times H$ (= the *Fruſtum*, &c.

Again,

Suppoſe	1	$x = D - d$. And F = the *Fruſtum*
Then	2	$DD + Dd + dd = \frac{3F}{H}$, by the 7th Step of the *laſt*
1 ⊖ 2	3	$xx = DD - 2Dd + dd$
2 — 3	4	$3Dd = \frac{3F}{H} - xx$
4 ÷ 3	5	$Dd = \frac{F}{H} - \frac{1}{3}xx$. Or $Dd + \frac{1}{3}xx = \frac{F}{H}$
5 × H	6	$Dd + \frac{1}{3}xx : \times H = F$ the *Fruſtum* $ab\,AB$.

Hence we have another Eaſie *Theorem* for finding the ſame *Fruſtum*.

THEOREM XVI.

To the Rectangle of the Sides of the two Baſes, add one Third part of the Square of their Difference; that Sum being multiply'd into the Height, will produce the Solidity.

Example. Let $D = 16$. $d = 12$. and $H = 9$. As before;

Then $Dd = 192$. $D - = 4 = x$. $\frac{1}{3}xx = \frac{4 \times 4}{3} = 5{,}3333$

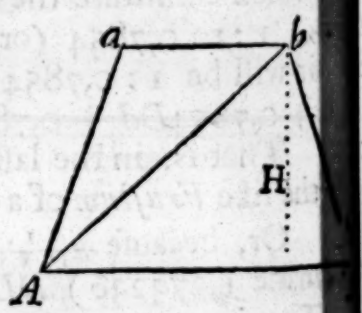
And $192 + 5{,}3333 = 197{,}3333$.

Laſtly $197{,}3333 \times 9 = 1775{,}9997$ the *Solidity* of the *Fruſtum* of the ſquare Pyramid. As before.

And $3{,}81968)\ 1775{,}9997\ (1394{,}87$, &c. the like *Fruſtum* of a Right Cone. As before.

Either of the two laſt *Theorems (being rightly apply'd)* will produce the true *Solid Content* of all *Fruſtums* of any kind of Pyramids, that are intercepted between two parallel and alike *Plains* or *Baſes*: As above.

But if ſuch *Fruſtums* are cut through the Extremities of both Baſes by a Diagonal Plain *(as A b in the annexed Figure)* into two parts, *A a b* and *A B b*, call'd *Hoofs*; then the *Solidity* of thoſe *Hoofs* is uſually found by dividing the middle Term *Dd* of the *Æquation* $DD + Dd + dd$ into Two parts, and adding one of thoſe parts to the *Square* of each Baſe.



Thus, $DD + \frac{1}{2}Dd : \times \frac{1}{3}H =$ the *Great Hoof A B b*.

And $dd + \frac{1}{2}Dd : \times \frac{1}{3}H =$ the *Leſſer Hoof A a b* of the *Fruſtum* of any ſquare Pyramid.

Then $3{,}8197)\ DD + \frac{1}{2}Dd : \times H$ (the *Greater Hoof* of a Cone.

And $3{,}8197)\ dd + \frac{1}{2}Dd : \times H$ (the *Leſſer Hoof*, &c.

Theſe are the *Theorems* made uſe of by Mr. *Darie*, in his Book of *Gauging*, and are pretty near the Truth, but not exactly ſo; for they give the *Solidity* of the upper Hoof *A a b* a ſmall matter too big, and the lower Hoof *A B b* as much too little.

Now, in order to rectifie that ſmall Error, I ſhall here propoſe the two following *Theorems*, which come very near the *Truth*, and are more eaſily perform'd than thoſe propos'd in the *firſt Impreſſion of this Book.*

Firſt,

First, $DD + \frac{1}{2}Dd + D - d : \times \frac{1}{3}H$ will be the *Solidity* of the *Greater Hoof A B b.*

Secondly, $dd + \frac{1}{2}Dd + d - D : \times \frac{1}{3}H$ will give the *Solidity* of the *Leſſer Hoof A a b*, of the *Fruſtum* of any *ſquare Pyramid.*

And for the like *Hoofs* of the *Fruſtum* of any *Right Cone*, it will be

Thus, 3,8197) $DD + \frac{1}{2}Dd + D - d : \times \frac{1}{3}H$ (= the *greater Hoof.*

And 3,8197) $dd + \frac{1}{2}Dd + d - D : \times \frac{1}{3}H$ (= the *leſſer Hoof.*

Note, *In order to avoid many Words in the following* Demonſtrations, *let* ⊙ *ſignifie any* Circle *in general; and if any two Letters be joyn'd to it, thus,* ⊙ *B A*, &c. *it then denotes the* Area *of ſuch a Circle as thoſe two Letters repreſent the* Radius *of.*

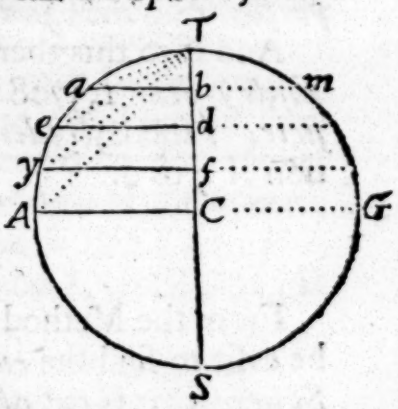
THEOREM XVII.

The Superficies of every Sphere (or Globe) *is equal to four times the* Area *of its Greateſt Circle.*

That is, of a Circle whoſe Diameter is the *Axis* of the *Sphere.*

Demonſtration.

If any *Semicircle (as A T G S)* be turn'd or mov'd about its Diameter *(T S)* it will deſcribe a ſolid Body call'd a *Sphere*, which will be conſtituted of an infinite ſeries of *concentrick* or *parallel Circles*, whoſe Diameters are Chords, *viz.* ⊙ *a b*, ⊙ *e d*, ⊙ *e f*, &c. by Definition 14.



Conſequently, the *Superficies* of the *Sphere* will be compos'd of the *Peripheries* of thoſe Circles which conſtitute its *Solidity.* By Definition 20.

Let $D = TS$, the *Axis* of any *Sphere.* Then, according to the Property of a Circle, it

will be	1	$D - Tb \times Tb = \square ab$
That is,	2	$D \times Tb - \square Tb = \square ab$
Therefore	3	$D \times Tb = \square aT$. For $\square ab + \square Tb = \square aT$
And	4	$D \times Td = eT$
	5	$D \times Tf = \square yT$, &c.

Hence

Hence 'tis evident, that the *Series* $\square aT$, $\square eT$, $\square yT$, &c. are in the ſame *Ratio* with Tb, Td, Tf, &c. *viz.* in *Arithmetick Progreſſion*. Whence it follows, that the $\odot aT$ = the Sum of all the Circle's *Peripheries* between T and b,

And $\odot eT$ = the Sum of all the Circle's *Peripheries* between T and d, &c.

Conſequently, that the $\odot AT$ = the Sum of all the Circle's *Peripheries* included between T and C; that is, $\odot AT$ = the *Superficies* of the *Semi-ſphere*.

And becauſe $\square AC + \square TC = \square AT$, and $\square AC = \square TC$ Therefore $\odot AT = 2 \odot AC$ is the *Superficies* of the *Hemi-ſphere*. Conſequently, $4 \odot AC$ will be the *Superficies* of the whole *Sphere*. Q. E. D.

Example. Suppoſe the Axis $TS = D = 16$. Then $DD = 256$. And $1 : 0{,}7854 :: 256 : 201{,}0624 = \cdot AC$. For $\frac{1}{4}D - AC$. Then $201{,}0624 \times 4 = 804{,}2496$, the *Superficies* of the whole *Sphere*.

Or, becauſe $3{,}1416$ is four times $0{,}7854$, therefore it will always be $1 : 3{,}1416 :: DD : 3{,}1416DD$ the *Superficies* of the *Sphere (as before)*; and it's equal to the *Curve Superficies* of a *Right Cylinder*, whoſe Diameter and Height are each $= D$ the Axis of the *Sphere*.

For $3{,}1416D$ = the *Periphery* of the *Cylinder*'s Baſe, and that multiply'd with D its Height, will be $3{,}1416DD$ the *Curve Superficies* of the *Cylinder*, by *Theorem* 12.

And if to this there be added the *Area* of its two Baſes, *(or Ends) viz.* $1{,}5708DD$, then 'tis evident, that the whole *Superficies* of the *Cylinder* will be to that of the *Sphere* in the Proportion of 3 to 2.

Scholium.

From the Method here uſed in proving the laſt *Theorem* 'twill be eaſie to find the *curve Superficies* of any *Segment* or *Part* of a *Sphere* that is *cut off* by a Right-line or Plain, *viz.* ſuch as the Segment aTm in the laſt *Scheme*, whoſe *curve Superficies* is $\odot aT$ *(as above)*. Therefore (becauſe $\square ab + \square Tb = \square aT$) it will be $\odot ab + \odot Tb =$ the *curve Superficies* of that *Segment*.

But if the Axis TS, and Height Tb, of the *Segment* are given, then it will be $TS \times Tb = \square aT$; as in the Third *Step* above. Which gives this Proportion or Theorem;

Viz.

Viz. { *As the* Axis *of the Sphere* : *Is to the whole* Superficies *of the Sphere* : : *So is the* Height *of any Segment* : *To its curve* Superficies.

To which if there be added the *Area* of the Segment's Baſe, the Sum will be the Superficies of the whole Segment.

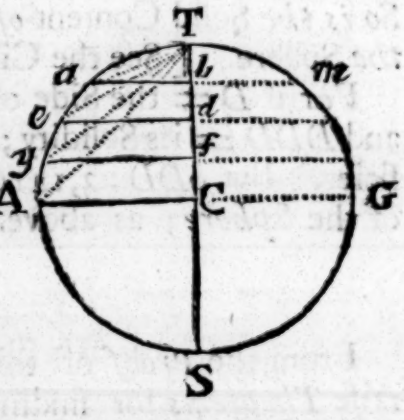
THEOREM XVIII.

Every Sphere is equal to two thirds of its circumſcribing Cylinder.

That is, of a Cylinder whoſe Height and Diameter of its Baſe are each equal to the Axis of the Sphere.

Demonſtration.

According to the Work in the laſt *Theorem* it appears, that ⊙ *ab*, ⊙ *ed*, ⊙ *yf*, &c. do conſtitute the Solidity of the Sphere; and, that □ *aT*, □ *eT*, □ *yT*, &c. are a ſeries of Terms in *Arithmetick* Progreſſion, □ *AT* being the *greateſt Term*, and *TC* the *Number* of *Terms*; Therefore ⊙ *AT* × ⅓ *TC* = the Sum of all the Series, per *Lemma* 2.



And becauſe □ *aT* − □ *Tb* = □ *ab*, □ *eT* − □ *Td* = *ed*, □ *yT* − □ *Tf* = □ *yd*, □ *AT* − □ *TC* = □ *AC*, &c. wherein □ *Tb*, □ *Td*, □ *Tf*, &c. are Series of Squares whoſe *Roots* *Tb*, *Td*, *Tf*, are in Arithmetick Progreſſion, □ *TC* being the *greateſt Term*, and *TC* the *Number* of *Terms*; Therefore ⊙ *TC* × ⅓ *TC* = the Sum of all that Series, *per* Lemma 3.

Conſequently, ⊙ *AT* × ½ *TC* : − ⊙ *TC* × ⅓ *TC* = the Sum of the *Series* ⊙ *ab*, ⊙ *ed*, ⊙ *yf*, &c. which conſtitute the Solidity of the half-Sphere *ATG*. Put *D* = 2 *TC* the Axis of the Sphere, Then ¼ *D* = ½ *TC*, and ⅛ *D* = ⅓ *TC*. And becauſe □ *AT* = 2 *TC*, therefore □ *AT* = 2 ⊙ *TC* = 1,5708*DD*. And 1,5708*DD* × ¼ *D* = 0,3927*DDD*.

Again, ⊙ *TC* × ⅓ *TC* = 0,7854*DD* × ⅛ *D* = 0,1309*DDD*.

Then 0,3927*DDD* − 0,1309*DDD* = 0,2618*DDD* the Solidity of the *Semi-ſphere ATG*.

Conſequently, 0,2618*DDD* × 2 = 0,5236*DDD* will be the ſolid Content of the whole *Sphere*, which is equal to two Thirds of the Cylinder whoſe Diameter of its *Baſe* and *Height* = *D*.

For 0,7854*DDD* = the Solidity of the Cylinder, by *Theorem* 11, but ⅔ of 0,7854*DDD* = 0,5236*DDD*; as before.

Therefore, &c. as by *Theorem*.

Exam-

Example. Suppoſe the *Axis* $D = 16$, then $DDD = 4096$, and $1 : 0{,}5236 :: 4096 : 2144{,}6656$ the *ſolid Content* of that *Sphere*.

Corollaries.

1. Hence it appears, that the ſolid Content of every Sphere is equal to its Superficies multiply'd into one Sixth part of its *Axis*.

For its Superficies is $3{,}1416DD$, by *Theorem* 17.

But $3{,}1416 \times \frac{1}{6} D = 0{,}5236DDD$ the ſolid Content, as before.

2. And hence 'tis alſo evident, that there is the like *Ratio* or Habitude between the *Cube* and its *inſcrib'd Sphere*, as is betwixt the *Square* and its *inſcrib'd Circle*; and that is, *As the* Superficies *of any* Cube : *Is to the* Superficies *of its* Inſcrib'd Sphere : : *So is the* Solid Content *of that* Cube : *To the* Solid Content *of the* Sphere. [See the Circle's Proportion, *page* 407.]

For if $D =$ the Side of the *Cube*, then $6DD =$ its Superficies, and $DDD =$ its Solidity; and $3{,}1416DD =$ the *Sphere*'s Superficies. But $6DD : 3{,}1416DD :: DDD : 0{,}5236DDD$ the Solidity of the *Sphere*; as above.

Scholium.

From the *Proof* of this *Theorem* 'twill be eaſie to deduce or raiſe *Theorems* for finding the Solid Content of any *Fruſtum* or *Segment* of a *Sphere*; as *a Tm* in the laſt Figure.

For we there ſuppoſe the Segment *a Tm* to be conſtituted of an *infinite ſeries* of Circles, which have the ſame *Ratio* with all thoſe Circles that conſtitute the *Semi-ſphere*.

Therefore it follows, that $\odot at \times \frac{1}{2} Tb : - \odot bT \times \frac{1}{3} Tb$ will be the Sum of all the Circles intercepted between *T* and *b*. Conſequently 'twill be the Solidity of that Segment.

And becauſe $\square ab + \square Tb = \square aT$: Therefore

$\odot ab + \odot Tb \times \frac{1}{2} Tb : - \odot Tb \times \frac{1}{3} b =$ the ſame *Solidity*.

Let $c = ab$ half the *Segment*'s Baſe; $h = Tb$ its *Height*; and $S =$ the *Solidity* of the *Segment* or *Fruſtum*:

Then $\odot ab = 3{,}1416cc$, and $\odot TB = 3{,}1416hh$.

Conſequently, $\frac{3{,}1416cch + 3{,}1416hhh}{2} - \frac{3{,}1416hhh}{3} = S$

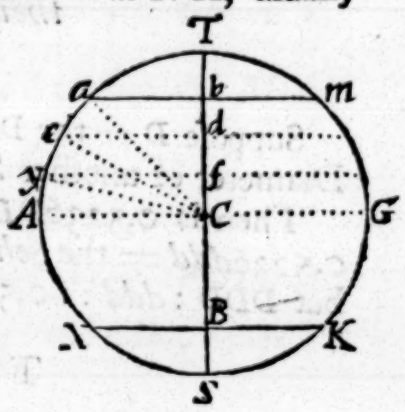
which being reduced, will become $3cch + hhh \times 0{,}5236 = S$.

Or $1{,}909855)\ 3cch + hhh (= S$. For $0{,}5236)\ 1{,}0000\ (1{,}909855$ which is one *Theorem* for finding the *Fruſtum*'s *Solidity*.

Note,

Note, Here we ſuppoſe the Height of the Segment, and the Diameter of its Baſe to be given; but if the *Axis* of the Sphere, and the Height of the Segment be given, then putting D = the Sphere's Axis, h = the Segment's Height, and c as before, 'twill be $D - h \times h = cc$, viz. $Dh - hh = cc$.
Therefore $3Dhh - 2hhh = 3cch - hhh$.
Conſequ. $3Dhh - 2hhh \times 0{,}5236 = S$ the *Fruſtum*'s Solidity.
Or $1{,}90985)\ 3Dhh - 2hhh\ (= S$. As before.
Which is a ſecond *Theorem* for finding the ſame *Fruſtum a Tm*.

And if it be requir'd to find the middle Part *a m NK*, uſually call'd the *middle Zone* of a Sphere, then becauſe 'tis ſuppos'd that *a m NK*, or which is all one, that $b C = CB$, therefore it is plain, that if *twice* the Segment *a Tm* be taken from the Solidity of the whole Sphere, there will remain the middle *Zone a m NK*.



But becauſe that Work is a little troubleſome, I ſhall here ſhew how to raiſe a Theorem for the doing it.

Firſt, Becauſe $AC = yC = eC = aC = TC$. Therefore it will be $\square AC - \square Cf = \square yf$. $\square AC - \square Cd = \square ed$. $\square AC - \square Cb = \square ab$, &c.

Here becauſe $\square AC$. $\square AC$. $\square AC$, &c. are a Series of *Equals*, and CB the Number of all the *Terms*, therefore $\square AC \times Cb$ = the Sum of all that Series, by *Lemma* 1.

And $\square Cf$. $\square Cd$. $\square Cb$. &c. being a Series of Squares whoſe *Roots* are in *Arithmetick Progreſſion*, beginning at the Centre or Point C, viz. $0, Cf, Cd, Cb$, &c. wherein the greateſt Term is $\square Cb$, and Number of Terms is Cb. *Ergo* $\square Cb \times \frac{1}{3} Cb$ = the Sum of all the Series, by *Lemma* 3.

Conſequently, the $\odot AC \times Cb : - \odot Cb \times \frac{1}{3} Cb$ = the Sum of all the Series, $\odot yf$. $\odot ed$. $\odot ab$, &c. which do conſtitute the Solidity of the *half-Zone amAG*.

And becauſe $\square AC - \square Cb = \square ab$. *Ergo* $\odot AC - \odot ab = \odot Cb$.
Conſ. $\odot AC \times Cb : - \frac{\odot AC + \odot ab : \times CB}{3} = 2 \odot AC + \odot ab : \times \frac{1}{3} Cb$
will be the Solidity of the *half-Zone*.

Put $D = AC = 2AC$. $x = am$. and $H = bB = 2Cb$.

Then $\odot AC = 0{,}7854DD$. $\odot ab = 0{,}7854xx$. And if we turn the common Factor 0,7854 into the Diviſor 1,27323

and then take *triple* of that Diviſor, *viz.* 3,8197 *(as before in the Fruſtums of Pyramids)* the *Reſult* of the precedent Work will produce this following Theorem.

THEOR. XIX. $\left\{\frac{2DD+xx}{3,8197}\right\} \times H = \left\{\begin{matrix}\text{the } \textit{middle Zone} \\ a\ m\ N\ K.\end{matrix}\right.$

THEOREM XX.

Spheres *are in Proportion one to another as the* Cubes *of their Diameters.* (18. *e.* 12.)

Demonſtration.

Suppoſe D = the *Diameter* or *Axis* of any *Sphere*, and d = the Diameter of another *Sphere*, either *greater* or *leſſer*.

Then is $0,5236DDD$ = the *Solidity* of one *Sphere*, and $0,5236ddd$ = the *Solidity* of the other *Sphere*, by *Theorem* 18; but $DDD : ddd :: 0,5236\,DDD : 0,5236ddd$. Q. E. D.

THEOREM XXI.

The ſolid Content of every Spheroid *is equal to Two thirds of its circumſcribing* Cylinder.

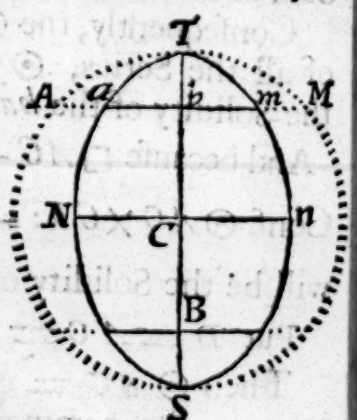
Demonſtration.

Suppoſe the Figure $NTn\text{-}S\,N$, in the annex'd *Scheme*, to repreſent a *Spheroid*, form'd by the Rotation of the *Semi-Ellipſis* TNS, about its *Tranſverſe Axis* TS, *(as by* Definition 15.)

Let $D = TS$, the *Length* of the *Spheroid*, and the *Axis* of its *circumſcribing Sphere*; and $d = Nn$, the Diameter of the *greateſt* Circle of the *Spheroid*.

Then becauſe $\square TC : \square NC :: \square Ab : \square ab$, by Step 3 in *Theor.* 7, Therefore it will be $DD : dd :: \square Ab : \square ab :: \odot Ab : \odot ab$, &c.

But the Sum of an *Infinite Series* of ſuch Circles as $\odot Ab$, *(whoſe* Diameters *are* Chords) do conſtitute the Solidity of the *Sphere*, *(as before at Theorem* 18*)* and the Sum of an *infinite ſeries* of ſuch Circles as $\odot ab$ (viz. *whoſe* Diameters *are Ordinates of the Ellipſis)* do conſtitute the Solidity of the *Spheroid*, by Definition 15.



Ergo $DD : dd :: 0,5236DDD : 0,5236ddD$ = the Solidity of the *Sphercid*, by *Lemma* 6.

But

But $0{,}5236ddD = \frac{1}{2}$ of the *Cylinder* whose Diameter is $= d$, and Height $= D$, by *Theorem* 11. Q. E. D.

Now from this *Proportion* between the *Sphere* and its *inscrib'd Spheroid*, 'twill be very easie to deduce *Theorems* for finding the *Solid Content* either of the *Segment* or *Middle Zone* of any *Spheroid*, having the same *Height* with that of the *Sphere*.

For { *As the Solidity of the whole Sphere : Is to the Solidity of the whole Spheroid : : So is any part of the Sphere : To the like part of the Spheroid*, by *Converse* to *Lemma* 6.

As for instance; Suppose it were requir'd to find the *Middle Zone* of any *Spheroid*:

Let $D = TS$, and $d = Na$, as above; and $H = bB$. $x = AM$, as in *Theorem* 19; and let $c = am$.

Then $\frac{2DD + xx}{3{,}8127} \times H =$ the *Middle Zone* of the *Sphere*. And

$0{,}5236DDD : 0{,}5236ddD : : \frac{2DD + xx}{3{,}8197} \times H : \frac{2dd \times H}{3{,}8197} + \frac{xx\,dd \times H}{3{,}8197DD}$

= the *Middle Zone* of the *Spheroid*.

Again, $DD : dd : : xx : cc$. Therefore $\frac{x\,x\,d\,d}{D\,D} = c\,c$.

Consequently, $\frac{xx\,dd}{DD} \times \frac{H}{3{,}8197} = \frac{cc}{3{,}8197} \times H$. Which being taken instead of $\frac{x\,x\,d\,d \times H}{3{,}8197DD}$ there will arise this following

THEOREM XXII. $\left\{ \frac{2\,d\,d + c\,c}{3{,}8197} : \times H = \right\}$ the *midd. Zone* of the *Spheroid* being the very same with *Theorem* 19.

Note, *In the same manner you may raise* Theorems *for finding the* Segment *of a* Spheroid, *cut off either of its Ends*, &c.

THEOREM XXIII.

The Area *of every* Parabola *is Equal to Two Thirds of its Circumscribing* Parallelogram.

Demonstration.

Let the Figure SAB represent half a *Parabola*; make DB parallel to the *Axis* SA, and Sd parallel to the *Semi-Ordinate* AB. And suppose Sd to be divided into an *infinite*

Series of *Equidiſtant Points*, as *f*, *g*, *h*, &c. and from thoſe *Points* imagine a Series of *parallel Lines*, viz. *f m*, *g n*, *h p*, &c. to touch the *Curve* of the *Parabola*, and meet the *Semi-ordinates* *m a*, *n e*, *y p*, &c.

Then, according to the *Property* of the *Parabola*, it will be

1 $SA : \square AB :: Sa : \square am$
2 $SA : \square AB :: Se : \square en$
3 $SA : \square AB :: Sy : \square yp$, &c.

But $Sa = fm$. $Se = gn$. $Sy = hp$. $SA = dB$

Therefore alternately it will be

3, 4 $\square AB : dB :: \square yp : hp$
2, 5 $\square AB : dB :: \square en : gn$
1, 6 $\square AB : dB :: \square am : fm$, &c.

In theſe *Proportions* $\square am$, $\square en$, $\square yp$, &c. are a *ſeries* of *Squares* whoſe *Roots* *Sf*, *Sg*, *Sh*, &c. are in *Arithmetick Progreſſion*, beginning at the Point *S*. And becauſe the Lines *h p*, *g n*, *f m*, &c. have the ſame *Ratio*, therefore they are as ſuch a *ſeries of Squares*, wherein *d B* is the *Greateſt Term*, and *S d* the *Number of Terms*.

Conſequently $\frac{db \times Sd}{3}$ = the *Sum* of all thoſe Lines, by *Lemma* 3.

But $SA \times AB = dB \times Sd$. Therefore $\frac{SA \times AB}{3}$ = the *Sum* of all that *ſeries of Lines*; but all thoſe *Lines* do conſtitute the *Area* of the *Semi-Parabola*'s *Complement*, viz. the *Area* of what *half* the *Parabola* wants of compleating or filling up the *Parallelogram* *S d A B*.

Wherefore $SA \times AB : -\frac{1}{3} SA \times AB = \frac{SA \times AB}{3}$ will be the *Area* of *half* the *Parabola* *S A B*.

Conſequently, $\frac{1}{3} SA \times bB$ will be the *Area* of the whole *Parabola* *b S B*. Q. E. D.

Example. Suppoſe the *Baſe*, or *greateſt Ordinate*, of a *Parabola* to be $bB = 24$, and its intercepted Diameter (or Axis) be $SA = 33$; Then $2SA \times bB = 66 \times 24 = 1584$. and 3) 1584 (528 the *Area* of that *Parabola*.

THEOREM. XXIV.

Every Parabolick Conoid is equal to one half of its Circumſcribing Cylinder.

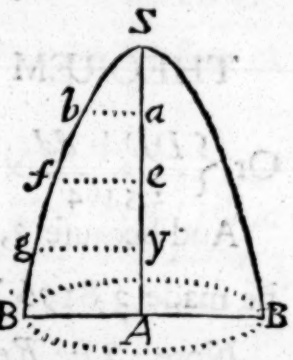
Demon-

Demonſtration.

If any *Semi-Parabola* (as *B S A*) be turn'd or moy'd about its *Axis*, (*S A*) 'twill form a *Solid Parabolick Conoid*, conſtituted of an *infinite Series* of *Circles*, viz. ⊙ *b a*, ⊙ *f e*, ⊙ *g y*, &c. by *Defin.* 17.

Now, according to the *Property* of every *Parabola*, it will be, $SA : AB :: AB : \frac{\square AB}{SA} = L$, the *Latus Rectum*.

Then $\begin{cases} Sa \times L = \square ba \\ Se \times L = \square fe \\ Sy \times L = \square gy, \text{ \&c.} \end{cases}$



Here $Sa \times L$, $Se \times L$, $Sy \times L$, &c. are a *Series* of *Terms* in Arithmetick Progreſſion; Therefore $\square ba$, $\square fe$, $\square gy$, &c. are alſo a *Series* of *Terms* in the ſame *Progreſſion*, beginning at the Point *S*, wherein $\square AB$ is the *greateſt Term*, and *SA* the *Number* of all the *Terms*. Therefore $\square AB \times \frac{1}{2} SA =$ the *Sum* of all the *Series*, by *Lemma* 2.

Conſequently, $\odot AB \times \frac{1}{2} SA =$ the *Sum* of all the *Series* of ⊙ *b a*, ⊙ *f e*, ⊙ *g y*, &c. which do conſtitute the *Solidity* of the *Conoid*.

And putting $D = 2AE$, and $H = SA$,

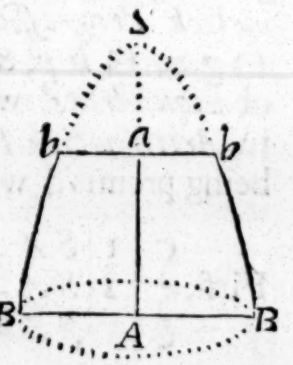
Then $0{,}7854DD \times \frac{1}{3}H = 0{,}3927DDH$ will be the *Solid Content* of the *Conoid*; which is juſt half the *Cylinder* whoſe *Baſe* $= D$ and *Height* $= H$. [See *Theorem* 11.] Q. E. D.

This being underſtood, 'twill be eaſie to raiſe a *Theorem* for finding the lower *Fruſtum* of any *Parabolick Conoid*.

For, ſuppoſing $h = aA$ the *Height* of the *Fruſtum*, and $p = Sa$ the *Height* of the Part *b S b* cut off; Then $h + p = SA$, the *Height* of the whole *Conoid*.

Conſequently, $\frac{\odot AB \times h + \odot AB \times p}{2}$ = the *Solidity* of the whole *Conoid*.

And $\frac{\odot ba + p}{2}$ = the *Solidity* of the Part cut off.



Ergo 1 $\left\{ \frac{\odot AB \times h + \odot AB \times p - \odot ba \times p}{2} = \right.$ the *Solidity* of the *Fruſtum*.

But 2 $h + p : \square AB :: p : \square ba$

Conſeq. 3 $h + p : \odot AB :: p : \odot ba$

3 4 $\odot AB \times p = \odot ba \times h + \odot ba \times p$

$4 - \odot ba \times p$	5	$\odot AB \times p : - \odot ba \times p = \odot ba \times h$
1×2	6	$\odot AB \times h : + \odot AB \times p : - \odot ba \times p = 2F$
$6 - 5$	7	$\odot AB \times h = 2F : - \odot ba \times h$
$7 + \odot ba \times h$	8	$\odot AB + h : + \odot ba \times b = 2F$
$8 \div 2$	9	$\frac{\odot AB + \odot ba}{2} : \times h = F$ the *Frustum*'s Solidity.

Let $D = AB$, as before; and $d = 2ba$, the Diameter of the *Part* cut off; Then we shall have this following

THEOREM XXV. $\left\{ 0{,}3927DD + 0{,}3927dd : \times h = \right.$ the *Solidity* of the *Frustum* requir'd.

Or $\left\{ \frac{DD + dd}{2{,}5464} \times h = \right.$ the *Frustum*; for ,3927) 1,0000 (= 2,5464

And because $2{,}5464 + \frac{2{,}5460}{2} = 3{,}8196$; Therefore it may be made 3,8196) $DD + dd : \times \frac{1}{2} h$ (= the same *Frustum*, &c.

Note, *The Reason why I have reduced this* Theorem *to have the same* Divisor *with those at the* Frustums *of Pyramids*, &c. *will best appear farther on*, viz. *when they all come to be apply'd to Practice in* Gauging.

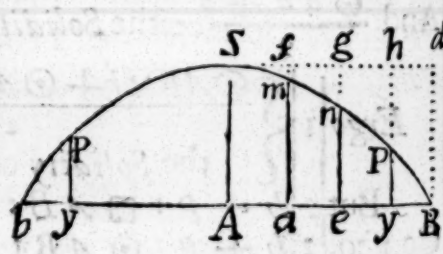
THEOREM XXVI.

Every Parabolick Spindle (or Pyramidoid) *is Equal to Eight Fifteenths of its circumscribing Cylinder.*

Demonstration.

If any *Acute Parabola*, as bSB, be turn'd or mov'd about its greatest Ordinate bAB, it will form a Solid call'd a *Parabolick Spindle*, constituted of an *infinite series* of $\odot ma$, $\odot ne$, $\odot py$, &c. by Definition 18.

Let us suppose the Line Sd parallel to AB, &c. (as at *Theorem* 23) then it hath already been prov'd, that the Lines fm, gn, hp, &c. are a *series* of *Squares* whose *Roots* are in *Arithmetick Progression*: Consequently their *Squares*, viz, $\square fm$, $\square gn$, $\square hp$, &c. will be a *Series* of *Biquadrats*, whose *Roots* will be in *Arithmetick Progression*; which being premis'd, we may proceed thus.



First, $\left\{ \begin{array}{l|l} 1 & SA - fm = ma \\ 2 & SA - gn = ne \\ 3 & SA - hp = py, \text{\&c.} \end{array} \right.$

1 ⊙ 2 | 4 | $\square SA - 2SA \times fm + \square fm = \square ma$
2 ⊙ 2 | 5 | $\square SA - 2SA \times gn + \square gn = \square ne$
3 ⊙ 2 | 6 | $\square SA - 2SA \times bp + \square bp = \square py$, &c.

1. In these *Æquations* the $\square SA$, $\square SA$, $\square SA$ being a *series* of *Equals*, and AB the *Number* of all the *Terms*; Therefore it will be $\square SA \times AB =$ the *Sum* of the *Series*, by *Lemma* 1.

2. Because fm, gn, bp, &c. are as a *Series* of *Squares* wherein SA is the *greatest Term*, and AB the *Number* of all the *Terms*; Therefore $\frac{2SA \times SA \times AB}{3} = \frac{2\square SA \times AB}{3}$ will be the *Sum* of all that *Series*, by *Lemma* 3.

3. And the $\square fm . \square gn . \square bp$, &c. will be a *series* of *Terms* in the *Ratio* of *Biquadrates*, as above, $\square dB = \square SA$ being the *greatest Term*, and AB the *Number* of all the *Terms*; therefore it will be $\frac{\square SA \times AB}{5} =$ the *Sum* of all the *Series*, by *Lemma* 5.

Whence it follows, that $\square SA \times AB - \frac{2\square SA \times AB}{3} + \frac{\square SA \times AB}{5}$ = the *Sum* of all the *series* of $\square ma$, $\square ne$, $\square py$, &c.

That is, $\frac{8\square SA \times AB}{15} =$ the *Sum* of all the *series* of $\square ma . \square ne . \square bp . \square dB .$ &c.

Consequently, $\frac{8 \odot SA \times AB}{15} =$ the *Sum* of all the *series* of $\odot ma . \odot ne . \odot py .$ &c. which do constitute the *Solidity* of half the *Spindle*, *viz.* of SAB.

Therefore putting $D = 2SA$, and $H = 2AB$, (viz. bAB) it will be $0{,}41888DDH =$ the *Solidity* of the whole *Parabolick Spindle* bSB, being $\frac{8}{15}$ of $0{,}7854DDH$, the Solidity of its *Circumscribing Cylinder*. Q. E. D.

From hence we may also raise a *Theorem* for finding the *Frustum* $SApy$ of the last Figure.

For $\odot SA$ being the greatest Term, $\odot py$ the least Term, and Ay the *Number* of all the Terms or Circles included between A and y,

Therefore | 1 | $\square SA - \frac{2SA \times bp}{3} + \frac{\square bp}{5} : \times Ay = z$ the *Sum* of all the *Series* $\square SA, \square ma, \square gn, \square py$.

1 × 3 | 2 | $3\square SA - 2SA \times bp + \frac{3\square bp}{5} : \times Ay = 3z$

$2 \div Ay$

$2 \div Ay$	3	$3\square SA - 2SA \times hp + \frac{3\square hp}{5} = \frac{3z}{Ay}$
But	4	$\square SA - 2SA \times hp = \square py - hp$ By 6th *Step.*
3 − 4	5	$2\square SA + \frac{3\square hp}{5} = \frac{3z}{Ay}\ \square py + \square hp$
$5 \pm$ &c.	6	$2\square SA + \square py - \frac{2}{3}\square hp = \frac{3z}{Ay}$
Conſeq.	7	$2 \odot SA + \odot py - \frac{2}{3} \odot hp : \times \frac{1}{3} Ay = z.$ the

Sum of all the *Series* of $\odot SA . \odot ma . \odot ne . \odot py$, which do conſtitute the *Solidity* of the *Fruſtum SApy*. Therefore putting $D = 2SA$ (as before) $G = 2py$, $x = 2hp$, and $H = Ay$, it will be $1{,}5708\,DD + 0{,}7854\,CG - 0{,}31416xx : \times \frac{1}{3} H =$ the *Fruſtum SApy*. And if we make $L = 2H$,

Then $1{,}5708\,DD + 0{,}7854\,CG - 0{,}31416\,xx : \times \frac{1}{3} L =$ the Double of that *Fruſtum*, being the *middle Zone*. And by turning theſe *Factors* into one common Diviſor, as in the *Fruſtum* of the *Conoid* at *Theorem* 25, *page* 430, there will ariſe this following *Theorem*.

THEOREM XXVII.

$\{3{,}8196)\ 2DD + CC - 0{,}4xx : \times L\ (=$ the *middle Zone* of a *Parabolick Spindle*.

It may be here expected that I ſhould now proceed to ſhew how the *Area* of any *Hyperbola*, and the *Contents* of ſuch *Solids* as may be form'd by the *Rotation* of that *Figure* about its *Axis*, &c. may be found; but becauſe thoſe Things cannot be exactly perform'd by any *certain* or *ſettled Theorems*, as theſe of the *Circle*, *Ellipſis*, and *Parabola* have been, I have therefore omitted them, and refer the Reader to Dr. *Wallis's Algebra*, Chap. 90, &c. or to the *Philoſoph. Tranſact.* Numb. 34. wherein he may find the *Method* of forming *Infinite Series* relating to the ſquaring of an *Hyperbola*, &c. which are too tedious to be *fully explain'd* and *demonſtrated* in this ſmall *Tract*, it being only intended as an *Introduction*, the which I ſhall here conclude.

FINIS.

AN APPENDIX OF *Practical Gauging.*

THE Art of *Gauging* is that Branch of the *Mathematicks* call'd *Stereometry*, or the *Meaſuring of Solids*, becauſe the Capacities or Contents of all ſorts of *Veſſels* uſed for *Liquors*, &c. are computed as tho' they were really *ſolid Bodies*; which any one that hath made himſelf Maſter of the 'foregoing Parts of this Treatiſe may eaſily underſtand, without any farther Directions.

However, becauſe 'tis not to be ſuppos'd that every one who deſigns to undertake the *Office* or *Imployment* of a *Gauger*, hath made ſo great a progreſs in *Mathematical Learning*. I have therefore preſented the young *Gauger* with this *Appendix*, wherein I have only inſerted ſuch *Rules* as are uſeful in *Gauging*, and have been already demonſtrated in this *Treatiſe*. But herein, I preſuppoſe that he hath acquir'd (*or if not, 'tis very neceſſary he ſhould acquire*) a competent *Knowledge* both in *Arithmetick* and *Geometry*: That is,

I. In *Arithmetick* he ſhould underſtand the principal Rules very well, eſpecially *Multiplication* and *Diviſion*, both in whole *Numbers* and *Decimal parts*, (*which may be eaſily learnt out of the* 2*d*, 3*d*, *and* 5*th Chapters of Part* 1.) that ſo he may be ready at computing the Contents of any Veſſel, and caſting up his Gauges by the *Pen* only, *viz.* without the help of thoſe Lines of Numbers upon *Sliding-Rules*, ſo much applauded, and but too much practis'd, which at beſt do but help to *Gueſs* at the Truth: I mean ſuch *Pocket-Rules* as are but nine Inches (or a Foot) long, whoſe *Radius* of the double Line of Numbers is not ſix Inches; and therefore the Graduations or Diviſions of thoſe Lines are ſo very cloſe, that they cannot be well diſtinguiſh'd. 'Tis true, when the Rules are made Two or Three Foot long, (*I had one of ſix Foot*) then they may be of ſome Uſe, eſpecially in ſmall Numbers; altho' even then the Operations may be much better (and almoſt as ſoon) done by the *Pen*: For indeed, the chief Uſe of Sliding-Rules is only in taking of Dimenſions, and for that purpoſe they are very convenient.

II. In *Geometry* the Gauger ſhould underſtand not only how to take *Dimenſions*, *(which is beſt learnt by Practice)* but alſo how to *divide* any irregular Figure or Superficies, as *Brewers Backs* or *Coolers*, &c. into the eaſieſt and feweſt regular Figures they will admit of, that ſo their *Area's* may be truly computed with the leaſt Trouble. And this may be learn'd *(with a little care and diligence)* out of the 1ſt, 2d, and 5th Chapters of Part III, which the Gauger ſhould be well acquainted with. Alſo he ought to have ſo much Skill in Solids, as to be able, even at ſight, *(but this muſt be acquir'd by Experience)* to determine what ſort of Figure any Veſſel is of, (*viz.* any Tun, or cloſe Cask) or what Figure it may be beſt reduced to, ſo that its *Dimenſions* may be truly taken, and the Content thereof computed with the leaſt *Error*. I ſay, *with the leaſt Error*, becauſe 'tis very difficult, if not impoſſible, to do it exactly; for there is not any Tun or Caſk, *&c.* ſo regularly made as by the Rules of *Art* 'tis requir'd to be.

III. Beſides the aforemention'd, the *young Gauger* muſt know, that all *Dimenſions* uſeful in Gauging are to be taken in *Inches*, and Decimal parts of an *Inch*; and if they are taken in any other Meaſures, as Feet, Yards, *&c.* thoſe Meaſures muſt be reduced to Inches, *(ſee* Sect. 4, pag. 42) becauſe the Contents of all ſorts of Veſſels *(taken notice of in Gauging)* are computed by the Standard Gallon of its Kind, whoſe Content is *known* to be a certain Number of *Cubick Inches*: That is, the *Beer* or *Ale* Gallon contains 282, the *Wine* 231, and the *Corn* Gallon 268,8 Cubick Inches. [*See the five Tables*, &c. *in Page* 34, 35, 36, *which I here ſuppoſe the Gauger to have learnt perfectly, by heart.*] Conſequently, if either the Superficial or Solid Content of any Veſſel, as *Back*, *Tun*, *Cask*, &c. be once computed in Cubick Inches, 'twill be eaſie to know how many Gallons, either of *Ale*, *Wine*, or *Corn*, that Veſſel will hold.

Note, I have here ſaid, the Superficial Content in Cubick Inches, which may ſeem to be very improper, according to the Definition given of a *Superficies* in Page 279; but you muſt know, that in the Buſineſs of Gauging, all *Superficies* or *Area's* are always underſtood to be *one Inch deep*, otherwiſe it could not be ſaid *(as in the Gauger's Language it is)* that the *Area* of ſuch a Back, or of ſuch a Circle, *&c.* is ſo many *Gallons*.

Theſe things being very well underſtood, the *young Gauger* will be fitly prepar'd to underſtand the following *Problems*, which are ſuch as have *(moſt of them)* been already propos'd in the 'foregoing parts of this Treatiſe, and *only are here apply'd to Practice*; and therefore I ſhall, for Brevity's ſake, often refer to thoſe *Theorems* and *Problems*.

Sect.

Sect. 1. *To find the* Area *of any right-lined* Superficies *in Gallons.*

PROBLEM I.

To find the Area *of any ſquare* Tun, Back, *or* Cooler, *&c. either in* Ale, Wine, *or* Corn *Gallons.*

Rule. *Multiply the given* Length *or* Breadth (being here equal) *into itſelf, and the Product will be the* Area *in Inches; then divide that* Area *by* 282, *or* 231, *or* 268,8 *and the Quotient will be the* Area *requir'd.*

Example. Suppoſe the Side of a ſquare Tun, Back, or Cooler be 124,5 Inches, what will its *Area* be in Gallons?

Firſt 124,5 × 124,5 = 15500,25 the *Area* in *Inches.*
Then 282) 15500,25 (54,96, *&c.* the *Area* in *Ale Gallons.*
And 231) 15500,25 (76,10, *&c.* the *Area* in *Wine Gallons.*
Or 268,8) 15500,25 (57,66, *&c.* the *Area* in *Corn Gallons.*

But if any one would rather work by *Multiplication* than by *Diviſion*, he may turn or change any *Diviſor* into a *Multiplicator*, if he *divide* Unity, or 1, by that *Diviſor.* (Vide *Probl.* 3, *pag.* 402.)

Thus 282) 1,000000 (0,003546 the Multiplicator for *Ale Gallons*
And 231) 1,000000 (0,004329 the Multiplicator for *W. Gallons.*
Or 268,8) 1,000000 (0,003722 the Multiplicator for *C. Gallons.*

Conſequently 15500,25 × 0,003546 = 54,96, *&c.* the *Area* in *Ale Gallons*; as before. And ſo on for the reſt.

PROBLEM II.

To find the Area *of any* Tun, Back, *or* Cooler *in the form of a Right-angled* Parallelogram, *in Ale Gallons*, &c.

See the Rule for finding its *Area* in *Inches*, at *Probl.* 1, *p.* 339, then either *divide (or multiply)* that *Area*, as above, and you will have the *Area* in Gallons.

Example. Suppoſe the *Length* of a Brewer's Tun, Back, or Cooler be 217,5 Inches, and its *Breadth* 85,6 Inches, what will its *Area* be in Ale or Beer Gallons, *&c.*?

Firſt 217,5 × 85,6 = 18648. Then 282) 18648 (66,12, *&c.*
Or 18648 × 0,003546 = 66,12, *&c.* the *Area* requir'd, *&c.*

PROBLEM III.

To find the Area *of any Triangular* Tun, Back, *or* Cooler, *in* Ale *Gallons*, &c.

See the Rule for finding its *Area* in Inches at *Prob.* 3, *p.* 340. then *divide (or multiply)* that *Area* as before, and you will have the *Area* requir'd.

Example. If the Length of the Baſe of a Triangular Cooler be 86,4 Inches, and its perpendicular Breadth be 57 Inches, what will its *Area* be in Ale Gallons?

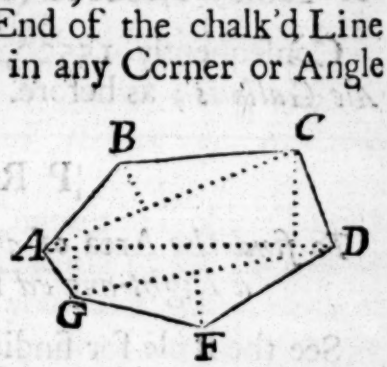
First, $86{,}4 \times \frac{57}{2} = 2462{,}4$. Then 282) 2462,4 (8,73, &c.

Or $2462{,}4 \times 0{,}003546 = 8{,}73$, &c. the *Area* in Ale Gallons.

Proceeding thus, you may eaſily find the *Area* of any Tun, Back, or Cooler, whether it be in the Form of a *Rhombus*, *Rhomboides*, *Trapezium*, or of any other *Polygon*, either regular or irregular, in Ale or Beer Gallons, &c. if you firſt *divide* it into *Triangles*, and then find the *Area's* of thoſe *Triangles*; *(as in the 2d, 4th, 5th, and 6th Problems in* Chap. 5, Part III.) *the* Sum of thoſe *Area's* being divided (or multiply'd) by its proper Diviſor (or Multiplicator) as above, will give the *Area* requir'd.

Now, the Practical Way of *dividing* any *Polygonous* Tun, Back, &c. into *Triangles*, is by help of a chalk'd Line, ſuch as the Carpenters uſe, and may be thus perform'd.

Suppoſe any Brewer's Tun, Back, or Cooler, in the Form of the annex'd Figure *A B C D F G*. Let one End of the chalk'd Line be faſten'd with a Nail *(or otherwiſe)* in any Corner or Angle of the Back, as at *A*, then ſtraining it to the Angle at *C*, ſtrike the Diagonal Line *A C* upon the Bottom of the Back; and ſtraining it again to the Angle *D*, ſtrike another Diagonal Line, as *A D*, and ſo on for the Diagonal Line *G D*, &c. Then having mark'd out all the Diagonals, the Perpendiculars may be thus found: Faſten (as before) one End of the chalk'd Line in the Angle *B*, and then by moving it *to* and *fro* upon the *ſtretch*, find out the neareſt Diſtance between the Angle at *B* and the Diagonal Line *A C*; there ſtrike a Line, and it will mark out the Perpendicular from *B* to the Line *A C*; and ſo on for the other Perpendiculars: Which being all mark'd out upon the Bottom of the Back, meaſure them and each Diagonal by a Line of *Inches*,



Inches, &c. and then the *Area* of that Back may be computed; as directed above.

And here, by the way, it may be obſerv'd, that the number of Triangles will always be leſs by Two, and the number of the Diagonals leſs by Three, than the number of the Sides *of any Right-lin'd Figure that is ſo divided.*

Having found (*as above*) the true *Area* of any Brewer's Back or Cooler, (*which, according to the Laws of Exciſe, ought always to be fix'd or immovable*) the next thing will be to find out the true *Dipping* or *Gauging Place* in that Back, that ſo the true *Quantity of Worts* may be computed (or caſt up) at any *Depth*; which may be thus done.

1. When the *bottom* of the Back is cover'd all over (*of any Depth*) either with Worts or Liquor, (viz. *Water*) then dip it in Eight or Ten ſeveral places (*more or leſs according to the largeneſs of the* Back) as remote and equally diſtant one from another as you well can, noting down the *Wet Inches* and *Decimal Parts* of every Dip.

2. Divide the Sum of all thoſe *Dips* or *wet Inches* by the Number of Places you dipp'd in, and the Quotient will be the *Mean Wet* of all thoſe Dips.

3. Laſtly, find out ſuch a place by the Side of the Back (*if you can*) that juſt wets the ſame with that *mean Dip*, and make a Notch or Mark there, for the true and conſtant *Dipping-place* of that Back. Then if any quantity of Worts (*which do cover the whole Back*) be dipp'd or gauged at that place, and the wet Inches ſo taken be multiply'd into the *Area* of the Back in *Gallons*, the *Product* will ſhew what *Quantity* (viz. *how many Gallons*) of Worts are in that Back at that time, provided the Sides of the Back do ſtand at *Right-angles* with its Bottom.

Sect. 2. *To find the* Area *of any* Circular *and* Elliptical Superficies *in Gallons.*

1. I have demonſtrated in *Chap.* 6, *Part* III, and *Theorem* 3; 5, 6. *Part* V. that the *Periphery* of the Circle whoſe Diameter is Unity, or 1, is 3,14159265, *&c.* (*or for common uſe* 3,1416) and that its *Area* is 0,78539816, *&c.* (or 0,7854 *fere.*)

2. Alſo, that the *Peripheries* of all Circles are in Proportion one to another as their *Diameters* are; and their *Area's* are in Proportion to the *Squares* of the *Diameters.* That is,

As 1 : 3,1416 : : the Diameter of any Circle : to its *Periphery.*
And 1 : 0,7854 : : the Square of the Diameter : to the *Area.*

Upon

Upon these Two Proportions depend the *Solution* of all the *Common* or *Practical Questions* about a Circle. [*See* Pag. 408,409.]

PROBLEM IV.

The Diameter *of any* Circle *being given in Inches, To find the* Periphery.

Rule. { *Multiply the given* Diameter *with* 3,1416, *and the* Product *will be the* Periphery *requir'd.* [See *Prob.* 1, *p.* 408]

Example. Suppose the Diameter of a Circle be 54,5 Inches, and it were requir'd to find its *Periphery*:

Then 54,5 × 3,1416 = 171,21, *&c.* Inches is the *Periphery* requir'd.

The *Converse* of this is easie, *viz.* by having the *Periphery* given, to find the *Diameter*. [See *Prob.* 3, *page* 408.]

PROBLEM V.

The Diameter *of any* Circle *being given,* (in Inches) *to find its* Area *in Gallons.*

Rule. { *Multiply the Square of the propos'd* Diameter *into* 0,7854, *and the Product will be the* Area *in Inches:* [see Probl. 2, p. 408] *That* Area *being divided by* 282, *or* 231, &c. *the Quotient will be the* Area *requir'd.*

Example. Suppose the given *Diameter* be 54,5 Inches, as above. First 54,5 × 54,5 = 2970,25. And 2970,25 × 0,7854 = 2332,83 the *Area* in Inches:

Then 282) 2332,83 (8,2724 the *Area* in *Ale* or *Beer* Gallons.
And 231) 2332,83 (10,0988 the *Area* in *Wine* Gallons.
Or 268,8) 2332,83 (8,6788 the *Area* in *Corn* Gallons.

But these *Area's* in Gallons may be much easier found, without knowing the Circle's *Area* in Inches, as above, by having the *Square* of the *Diameter* of that Circle whose *Area* is one Gallon; which may be thus found, by *Theorem* 6, *page* 407.

0,785398 : 1 : : 282 : 359,05 the *Square* of the *Diameter* of the Circle whose *Area* is 282 *cubick Inches*, viz. one *Ale* Gallon.

And from this Proportion will arise these following *Divisors*;
Viz. 0,785398) 282,000000 (359,05 will be a *Divisor* for *A. G.*
And 0,785398) 231,000000 (294,12 will be a *Divisor* for *W. G.*
Or 0,785398) 268,800000 (342,24 will be a *Divisor* for *C. G.*

IF

If the *Square* of the *Diameter* of any Circle be divided by any one of these constant or fixed *Divisors*, the *Quotient* will shew that Circle's *Area* in their respective Gallons: As for instance, in the last Circle, whose Square of its Diameter is 2970,25.

Then 359,05) 2970,25 (8,2725 the *Area* in *A. G.*
And 294,12) 2970,25 (10,0988 the *Area* in *W. G.* } As before.
Or 342,24) 2970,25 (8,6788 the *Area* in *C. G.*

Now these *Divisors* may be turn'd into *Multiplicators* by dividing Unity or 1, as in *page* 435: Or rather by dividing the *Area* in Inches of that Circle whose Diameter is 1.

That is, 0,785398 by 282. Or by 231, &c.

Thus 282) 0,785398 (0,002785 the *Multiplicator* for *Ale Gal.*
And 231) 0,785398 (0,003399 the *Multiplicator* for *Wine Gal.*
Or 268,8) 0,785398 (0,002922 the *Multiplicator* for *Corn Gal.*

These *Multiplicators* are the respective *Area's* of a Circle whose Diameter is 1; and therefore, if the Square of the Diameter of any Circle be *multiply'd* with any of these *Numbers*, the Product will be that Circle's *Area* in Gallons of the same Name:

Viz. 2970,25 × 0,002785 = 8,2725 the *Area* in *A. G.* as above.
And 2970,25 × 0,003399 = 10,0988 the *Area* in *W. Gal.* &c.

Thus you see, that if the Diameter of any Circle be given in Inches, there are Three several Ways of finding its *Area* in Gallons, and all equally true; but that which is perform'd by the constant Divisors is most generally practis'd.

PROBLEM VI.

The Transverse (or longest Diameter) *and the Conjugate* (or shortest Diameter) *of any* Elliptical Superficies *being given, To find its* Area *in Gallons.*

Rule. *Multiply the Two* Diameters (*viz.* the Length and Breadth) *together, and divide their Product by* 359,05 *for* Ale Gallons, *or* 294,12 *for* Wine Gallons, &c. *the Quotient will be the* Area *requir'd.* [See *Theorem* 7, *Page* 412.]

Example. Suppose the longest Diameter to be 73,5 Inches, and the shortest Diameter to be 51,6 Inches; What will the *Area* be in *Ale* Gallons?

First 73,5 × 51,6 = 3792,6. Then 359,05) 3792,6 (10,56 the *Area* in *Ale* Gallons. Or 294,12) 3792,6 (12,89 the *Area* in *Wine* Gallons, &c.

Note,

Note, The two laſt Problems are of great Uſe in gauging of *Worts* amongſt Country Victuallers, who generally brew but ſhort Lengths of Ale, *(perhaps between 20 and 60 Gallons at a Brewing)* and cool their Worts in ſeveral ſmall open Veſſels or Tubbs, whoſe Baſes or Bottoms are either a *Circle*, or an *Ellipſis*, having their Sides but low, and are moſt commonly *wider* at the Top than at the Bottom.

Now a practical Way of computing the *Quantity* of Worts that are at any time in one of thoſe *open Tubbs*, is briefly thus: When the *Tubb* is *dry*, find the true *Area* of its *Bottom* according to its Figure, *(as above)* and either mark that *Area* on the outſide of the *Tubb*, *(which was the way I generally us'd to order, becauſe the* Victuallers *did often lend their Cooling-Tubbs one to another)* or elſe *Number* the *Tubb*, and enter its *Area (and its Number)* into the *Stock-book*; then, when any of thoſe *Tubbs* hath Worts in it, take the Diameter of the *Surface* or *Top* of the Worts, and find that *Area*, adding *it* and the *Bottom Area* together. If either the *half-Sum* of thoſe two *Area's* be multiply'd with the Depth of the Worts, *(taken as near the Middle of the Tubb as you well can)* or, if the *Sum* of thoſe two *Area's* be multiply'd with half the Depth *(ſo taken)* the Product will ſhew the *Quantity* of thoſe Worts very near the Truth.

PROBLEM VII.

The Diameter *of any* Circle, *and the verſed Sine (viz.* the Height) *of any Segment, being given, To find the* Area *of that Segment in* Gallons.

In the 410th and 412th Pages you have Two Ways *(and their Examples)* of finding the *Area* of any *Segment* of a Circle in Inches; then if that *Area* in Inches be divided by 282, or 231, *&c.* the *Quotient* will be its *Area* in Gallons. But becauſe the *Area* of any ſuch *Segment* may be readily found in Gallons *(without finding its* Area *in Inches)* by help of a *Table* of *Segments*, whoſe Conſtruction is laid down in the Problem, *page* 411, &c. I have here inſerted a *Compendium* of ſuch a *Table*, which will ſerve very well for common Practice, not only to find the *Area* of any Segment of a Circle in Gallons, but alſo to find the Number of Gallons that are either *drawn out*, or remaining in any *Cylindrick Veſſel* lying along; or of any cloſe Cask, *(being firſt reduced to a* Cylinder*)* its *Axis* lying parallel to the *Horizon*, uſually call'd the *Ullage* of a Caſk; as ſhall be ſhew'd farther on.

A Table

A Table of the Segments of a Circle whose Area is Unity or 1, *the Diameter being divided by parallel Chord-Lines into* 100 *equal Parts.*

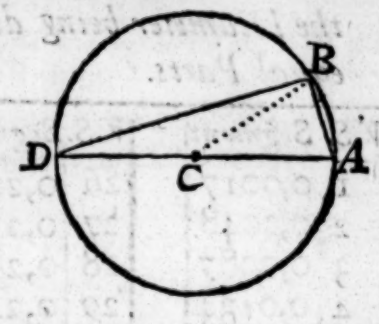
V.S.	Segment	V.S.	Segment	V.S.	Segment	V.S.	Segment
1	0,0017	26	0,2066	51	0,5127	76	0,8155
2	0,0048	27	0,2178	52	0,5255	77	0,8262
3	0,0087	28	0,2292	53	0,5382	78	0,8369
4	0,0134	29	0,2407	54	0,5509	79	0,8474
5	0,0187	30	0,2523	55	0,5635	80	0,8576
6	0,0245	31	0,2640	56	0,5762	81	0,8677
7	0,0308	32	0,2759	57	0,5888	82	0,8776
8	0,0375	33	0,2878	58	0,6014	83	0,8873
9	0,0446	34	0,2998	59	0,6140	84	0,8968
10	0,0520	35	0,3119	60	0,6265	85	0,9059
11	0,0598	36	0,3241	61	0,6389	86	0,9149
12	0,0680	37	0,3364	62	0,6514	87	0,9236
13	0,0764	38	0,3486	63	0,6636	88	0,9320
14	0,0851	39	0,3611	64	0,6759	89	0,9402
15	0,0941	40	0,3735	65	0,6881	90	0,9480
16	0,1032	41	0,3860	66	0,7002	91	0,9554
17	0,1127	42	0,3986	67	0,7122	92	0,9625
18	0,1224	43	0,4112	68	0,7241	93	0,9692
19	0,1323	44	0,4238	69	0,7360	94	0,9755
20	0,1424	45	0,4365	70	0,7477	95	0,9813
21	0,1526	46	0,4491	71	0,7593	96	0,9866
22	0,1631	47	0,4618	72	0,7708	97	0,9913
23	0,1738	48	0,4745	73	0,7822	98	0,9952
24	0,1845	49	0,4873	74	0,7934	99	0,9983
25	0,1955	50	0,5000	75	0,8045	100	1,0000

The Use of this Table of *Segments* depends upon the following Proportion,

viz. { *As the Diameter of any propos'd Circle* : *Is to* 100 (the Diameter of the Tabular Circle) : : *So is the Height of any Segment of the propos'd Circle* : *To a verſed Sine in the Table.*

Then if the *Tabular Segment*, which ſtands againſt that *verſed Sine*, be multiply'd into the Circle's *Area*, *(either in Inches or Gallons)* the Product will be the *Area* of the *Segment* requir'd, *(of the ſame Name)* viz. If the Circle's *Area* be Inches, the *Segment* will be *Inches*; if *Gallons*, the *Segment* will be *Gallons*.

Example. Let the Diameter of the given Circle be $DA = 62,5$ *Inches*, and the Height of the Segment ſought be $FA = 20$ *Inches*; What will its *Area* be in *Ale Gallons*?



Firſt, the *Area* of the whole Circle will be 10,8793 *Ale Gallons*, (by *Problem* 5.) and the *Proportion* will ſtand thus, 62,5 : 100 : : 20 : 32 the *verſed Sine* of the Table whoſe Segment is 0,2759.

Then $10,8793 \times 0,2759 = 3,0016$ *Ale Gallons*, being the *Area* of the Segment *BAGF*, as was requir'd. The like may be done for *Wine Gallons*, *Corn Gallons*, or *Inches*.

And, upon occaſion, the like Segments of any *Ellipſis* may be eaſily found. See the *Proportions* in the Corollaries to the 7th and 8th *Theorems*, *page* 412, &c. to which I here, for Brevity's ſake, refer the Reader.

Sect. 3. *To compute the Contents of ſuch Veſſels* (*viz.* Tuns, &c.) *as are in the Form of the following* Solids.

Note, Before the young *Gauger* proceeds to theſe Computations, he ſhould be well acquainted with ſuch Solids as are defin'd in *p.* 402 & 403, and then he may eaſily underſtand what ſort of *Figures* are meant in the following *Problems*, without the repetition of many Words.

PROBLEM VIII.

To find the Content of any Priſm *whoſe Sides are* Parallelograms, *what form ſoever its Baſe is of.*

That is, to compute the Content (in Gallons) of any *Tun*, &c. whoſe Sides are *Parallelograms* which ſtand upright, or at Right-angles with its Bottom.

Firſt find its ſolid Content in Inches, by *Theorem* 9, *page* 414; then divide that Content by 282, or 231, or by 268,8; the Quotient will ſhew the Content in their reſpective Gallons, *viz.* in *Ale*, *Wine*, or *Corn* Gallons.

Or elſe multiply the Content in *Inches* with 0,003546, or 0,004329, &c. [See the *Multiplicators*, page 435] thoſe Products will be the Content in their reſpective Gallons.

Or otherwiſe thus:

Find the true *Area* of the *Tun*'s Baſe or Bottom, as directed in *Sect.* 1, *p.* 435; that *Area* being multiply'd with the *Tun*'s Height (*viz. Depth within*) will produce the Content in Gallons; as before.

I take

I take the Work of this *Problem* to be ſo very eaſie, it needs no *Example*.

PROBLEM IX.

To find the Content of any Pyramid (in Gallons) *whoſe Baſe is bounded with Right-lines.*

Every Pyramid is *one Third part* of its *circumſcribing Priſm*; by *Theorem* 10, *page* 415. Therefore,

If the *Area* of the Baſe of any Pyramid, in *Gallons*, be multiply'd into *One Third* of its perpendicular *Height*; or if *one Third* of that *Area* be multiply'd with the whole *Height*, either of thoſe Products will be the Content of the Pyramid in Gallons, *&c.*

But the Content of any *ſquare Pyramid* may be eaſily found in *Gallons* by this *Rule*:

Rule. *Square the Side of its Baſe, and multiply that Square with the perpendicular Height; then divide that Product by* 846 = 282 × 3 *for Ale Gallons, or by* 693 = 231 × 3 *for Wine Gallons, or by* 806,4 = 268,8 × 3 *for Corn Gallons, the Quotient will be the Content requir'd.*

Or, if you multiply the ſaid Product with 0,001182 for *A. G.* or with 0,001443 for *W. G.* or, laſtly, with 0,001241 for *C. G.* the Reſult will be the Content requir'd; As before.

PROBLEM X.

To find the Content (in Gallons) *of the* Fruſtum *of any ſquare Pyramid, cut off by a Plain parallel to its Baſe.*

Firſt, Either by *Theorem* 15, *page* 419. *Theorem* 16, *p.* 420, find the propos'd *Fruſtum*'s Solidity in Cubick Inches; then divide that Content in Cubick Inches by 282 or 231, *&c.* and the *Quotient* will be the Content of the *Fruſtum* in their reſpective Gallons.

But, from the aforeſaid *Theorem* 15, there may be eaſily deduced the following *General Rule* for finding the Content of the like *Fruſtum* of any Pyramid, what Form ſoever its Baſes are of, *(ſuppoſing them to be parallel)* whether they are *alike* or *unlike*.

Rule. *Firſt find the* Area *of each Baſe*, (*viz.* the top and bottom *Area's* of the propos'd *Fruſtum*) *then find a Geometrical Mean between thoſe two* Area's (by Lemma 1, page 83) *the Sum of thoſe two* Area's *and their Mean being multiply'd into one Third of the* Fruſtum's *height, will produce the Content requir'd.*

Example. Suppoſe a *Tun* in the Form of the lower *Fruſtum* of a Pyramid, whoſe Baſes are *Equilateral Triangles:* Let the Side of the *Top* be 42 Inches, the Side of the *Bottom* be 63,4 Inches, and its *Height* (viz. *Depth*) be 33 Inches; What will the *Content* of that *Tun* be in *Ale Gallons*?

Firſt find the *Area* of that Baſe in Inches, by *Probl.* 7, *p.* 343; then find what thoſe *Area's* are in *Ale Gallons*, by *Probl.* 3, *p.* 436. *Multiply* thoſe two *Area's* together, the *ſquare Root* of their Product will be the *Mean Area*, &c. As in this *Example:*

Example. { The *Area* of the *Top* is 2,71 / The *Area* of the *Bottom* is 6,12 } *Ale Gallons.* / The *Mean Area* will be 4,07

Their *Sum* is 12,90

Then $12,9 \times \frac{23}{3} = 141,9$. Or $\frac{12,9}{3} \times 33 = 141,9$ the *Content* requir'd.

PROBLEM XI.

To find the Content of any Right Cylinder in Gallons.

That is, to compute the Content of any round *Tun*, &c. whoſe Diameters at *Top* and *Bottom* are equal, and at Right-angles with its *Sides.*

The Content of ſuch a *Tun* may be found by *Theorem* 11, *page* 415; or otherwiſe by the following *Rule.*

Rule. { *Multiply the Square of the* Diameter *into the Height, and divide the* Product *by* 359,05 (or multiply with 0,002785) *&c. as in Page* 439, *that Quotient* (or Product) *will be the Content requir'd.*

Exam. Suppoſe the Diameter be 42,5 and the Height 31,5 *Inches.* Firſt $42,5 \times 42,5 = 1806,25$. And $1806,25 \times 31,5 = 56896,875$ Then 359,05) 56896,875 (158,46 the Content in *Ale Gal.* &c.

PROBLEM XII.

To find the Content of any Cone *or* round Pyramid *in Gallons.*

Becauſe every Cone is one Third of its *circumſcribing Cylinder.* [*See* Theorem 13, page 416] therefore its Content may be truly found by the following Rule.

Rule. { *Multiply the Square of the* Diameter *of its Baſe into the perpendicular Height, then divide their Product by* 1077,15 = 359,05 × 3 *for Ale Gallons, or by* 882,36 = 294,12 × 3 *for Wine Gallons,* &c. *and the Quotient will be the Content requir'd.*

Or

Or if the ſaid *Product* be multiply'd with 0,000928 $= \frac{0,002785}{3}$ or with 0,001133 $= \frac{0,0034}{3}$ thoſe *Products* will be the Content in their reſpective Gallons.

Example. Suppoſe the Diameter of the *Baſe* be 42,5, and the perpendicular Height be 31,5 *Inches*, What will the Content be in *Ale Gallons?* (as before.

Firſt 42,5 × 42,5 = 1806,25. And 1806,25 × 31,5 = 56896,875 Then 1077,15) 56896,875 (52,82. Or 56896,25 × 0,000928 = 52,82 the *Content* in *Ale* Gallons: And ſo on for *Wine* or *Corn* Gallons.

PROBLEM XIII.

To find the Content of the lower Fruſtum *of any Cone in Gallons.*

That is, to compute the Content of any *round Tun*, &c. whoſe Diameters at *Top* and *Bottom* are parallel, but unequal.

The Content of ſuch a *Tun* may be found by the *Rule* at *Problem* 10; but from *Theorem* 16, *page* 420. 'twill be eaſie to deduce this following *Rule*.

Rule { *To the triple Product of the top and bottom* Diameters *add the Square of their difference*; *Multiply that Sum into the Height* (or Depth); *Then divide the laſt Product by* 1077,15 *for* Ale Gallons, *or by* 882,36 *for* Wine Gallons, *the Quotient will be the Content requir'd.*

Example. Suppoſe the Diameter at the *top* to be 52,4 *Inches*, the Diameter at the *bottom* 45,6, and the Height 30 *Inches*.

Firſt, 52,4 × 44,6 = 2337,04. And 2337,04 × 3 = 7011,12 } Add
Alſo 52,4 − 44,6 = 7,8 And 7,8 × 7,8 = 60,84 }
The Height 30 × 7071,96 = 212158,8

Then 1077,15) 212158,8 (196,96 } the *Content* in *Ale* Gall.
Or 212158,8 × 0,000928 = 196,96 }

And ſo on for either *Wine* or *Corn* Gallons, as occaſion requires. But if the *Tun (or Veſſel)* be not truly circular, that is, if either its *top* or *bottom (or both of 'em)* be Elliptical, whether they are *alike* or *unlike*, it matters not, the Content of ſuch a *Tun* may be truly found by the *General Rule* at *Problem* 10.

PROBLEM XIV.

The Axis *or* Diameter *of any Sphere or Globe being given in* Inches, *To find its Content in* Gallons.

Every *Sphere* is Two thirds of its *circumſcribing Cylinder*, by *Theor.* 18, *page* 423; from whence and *Theor.* 20, *page* 426, 'tis prov'd,

prov'd, that if the Cube of the *Axis* of any *Sphere (taken in Inches)* be multiply'd into 0,5236, the *Product* will be the Content of that *Sphere* in *Inches*. Consequently, if that Content be divided by 282, or by 231, *&c.* the *Quotient* will be the Content in *Gallons*.

But those two Works of multiplying with 0,5236, and then dividing by 282, or by 231, *&c.* may be contracted into one,

Thus 282) 0,5236 (0,001856 will be a *Multiplicat.* for *A.G.*
And 231) 0,5236 (0,002266 will be a *Multiplicat.* for *W.G.*
Or 0,5236) 282 (538,57 will be a *Divisor* for *Ale* Gallons.
And 0,5236) 231 (441,17 will be a *Divisor* for *Wine* Gallons.

From hence arises this following *Rule*.

Rule { *If the Cube of the* Axis *of any* Sphere *be divided by* 538,57 (or multiply'd with 0,001856) *or divided by* 441,17, (or else multiply'd with 0,002266) *the Quotient* (or Product) *will be the* Sphere's *Content in their respective* Gallons.

Example. Suppose the *Axis* or *Diameter* of a Sphere or Globe be 22 *Inches*, how many *Ale* Gallons may it hold?
Then 22 × 22 × 22 = 10648. And 538,57) 10648 (19,76 *A.G.*
Or 10648 × 0,001856 = 19,76 *Ale* Gal. the Content requir'd.
And so for either *Wine* or *Corn* Gallons, as Occasion requires.

PROBLEM XV.

To find the Content of any Segment *of a* Sphere *in Gallons.*

In the *Scholium*, *p.* 424, there are two *Theorems* for resolving this *Problem* according to the *Data*.

1. If the Diameter of the Segment's Base, and its Height, are given, the Content may be found by the first of those *Theorems*, which gives this *Rule*:

Rule 1. { *To the triple Square of half the Diameter add the Square of the Height, then multiply that Sum into the Height, and divide the Product by* 538,57 *for A.G. or by* 441,17 *for W.G.* &c. *as above.*

2. But if the *Axis* of the Sphere, and the *Height* of the Segment are given, the Content may be found by the *second* of those *Theorems*.

Rule 2. { *From the triple Product of the* Axis *into the Height substract twice the Square of the Height, then multiply the Remainder into the Height, and divide that Product by* 538,57, &c. *as in the last Problem.*

Either

Either of these Rules will produce the Content of the Segment in Gallons.

Example. Suppose the Diameter of the Segment's Base be 28 *Inches*, and its Height be 8 *Inches*, what may it contain in *Ale* Gallons.

First 2) 28 (14. Then (by *Rule* 1.) $14 \times 14 \times 3 = 588$. And $6 \times 6 = 36$. Next $588 + 36 = 624$. Again $624 \times 6 = 3744$. Lastly, 538,57) 3744 (6,95 the Content requir'd.

Note, *This Problem may be of Use in Gauging the Crowns of Brewers* Coppers, *&c.*

Sect. 4. *The practical Method of Gauging any fix'd* Tun *or* Copper, *and making a Table to shew what it will hold at every Inch deep, usually call'd* Inching of a Tun, *&c.*

First, you must know, that most (if not all) *Brewers Tuns* are so fix'd as to lean a little for conveniency of cleansing their Drink, which is usually call'd the *Drip* or *Fall of the Tun.* Now this Drip or Fall of any Tun is the *Hoof* of such a *Solid* as that *Tun* is suppos'd to represent, and under that Consideration it may be found, as in *Theor.* 16, *p.* 420: But the practical (*and indeed the best*) way is, to *measure* into the *Tun* (*when 'tis dry*) so much *Liquor* as will just cover its *Bottom*; for by that means you do not only find the true Fall, but also a true horizontal or level Plain over the *Bottom* of the *Tun*, from which if the *Depth* of the *Tun*, (viz. *the neareſt Diſtance from the top of the Tun to the ſurface of the Liquor*) be *ſet off* upon every one of its Sides, you will then have a true parallel Plain at the *top* of the *Tun* to that of the Liquor. Then, if the Sides of the *Tun* are *ſtreight* from the *top* to the bottom, take as many Dimensions in the *aforeſaid* two Plains as are needful to find the true *Area* of each; and by those *two Area's* and the *aforeſaid Depth* find so much of the Tun's Content (*by the General Rule at Problem* X.) as is betwixt those *two Plains.*

Next, to *Inch* that Tun, *divide the Difference* between the *top* and *bottom Area's* by the *aforeſaid Depth*, and the *Quotient* will be an *Addend* or *fix'd Number*; which being added to the lesser *Area*, the *ſum* will be the *Area* of the next *Inch*; and being added to that *Area*, their *ſum* will be the *Area* of the Third *Inch*; and so on from *Inch* to *Inch*, until the *Area* of every single *Inch* be found; the *ſum* of those *Area's* (*if the Work be true*) will amount (*or be equal*) to the Content found, as above. And if the

the *Tun*'s *Drip* or *Fall* be added to the ſum of all thoſe *Area*'s, that *ſum* will be the whole or full Content of that *Tun*.

Now from hence it muſt needs be eaſie to conceive, that if 1. 2. 3. or any *Number* of thoſe *Area*'s accounted from the *Bottom*, be added to the *Fall*, that *Sum* will ſhew the *Quantity* of *Liquor* or *Drink* that is in the *Tun*, to ſuch a Number of *wet Inches* from the *Bottom* as there were *Area*'s added together.

Or, if the *Sum* of any *Number* of thoſe *Area*'s *(being accounted from the Top)* be ſubſtracted from the *Tun*'s whole *Content*, the Remainder will ſhew what *Quantity* of *Liquor* or *Drink* is in the *Tun*, when there is ſuch a number of *dry Inches* from the *Top* as there were *Area*'s ſubſtracted.

This being well conſider'd, it will be eaſie to make a *Table* either to every *wet* or *dry Inch* of any *Regular Tun*, (viz. *whoſe Sides are ſtreight from top to bottom)* what Form ſoever its *Baſes* are of, and whether it ſtand upon the greater or leſſer *Baſe*.

But if the *Sides* of the *Tun* are irregular, (viz. *not ſtreight from its Top to the Bottom)* then the beſt and eaſieſt way will be to divide or part the *Tun* into ſeveral *Fruſtums*, each of *ten Inches* deep; and finding the Content of every ſingle *Fruſtum*, by taking the Diameters in the middle of every one of thoſe *ten Inches*, (*that is, the firſt Diameter at* 5 *Inches from the Top*; *the ſecond Diameter at* 15 *Inches from the Top*, &c.) and multiplying their reſpective *Area*'s with 10, *(which is done by only removing the ſeparating Comma's one place forward to the right Hand)* if the *Sum* of all thoſe *Fruſtums* be added to the *Fall*, *(as before)* that *Sum* will be the whole Content of the *Tun*.

Note, *If you take the Height of the foreſaid ten Inch* Fruſtums *in the Side of the* Tun, *you muſt allow for the Difference between the ſlant Height and the perpendicular Height in every* Fruſtum.

Laſtly, If from the whole Content of the Tun you ſubſtract the *Mean Area* of the firſt *Fruſtum* ten times, and from the Remainder ſubſtract the *Mean Area* of the ſecond *Fruſtum* ten times, and from the laſt Remainder ſubſtract the *Mean Area* of the third *Fruſtum*, &c. until there remain nothing but the *Fall* or *Hoof* of the *Tun*, you will then by that means have a Table that will ſhew what *Quantity* of *Drink* is in the *Tun* to any number of *dry Inches*.

And this is alſo the *Method* of *Gauging* and *Inching* Brewers Coppers, *viz.* by firſt meaſuring into the Copper ſo much *Liquor* as will juſt cover its *Crown*, and then dividing its perpendicular Height into *Fruſtums*, and its Sides into *four equal parts*, that ſo *croſs Diameters* may be taken in the *middle* of each *Fruſtum*.

But

But if the *Copper* be much *wider* at the *Top* than at the *Bottom*, and its *Sides ſpheroidal* or *arching*, as generally all *Large Coppers* are; then, inſtead of taking thoſe *Mean Diameters* in the *middle* of every *Ten Inches*, as above, you muſt take them in the *middle* of every *Six Inches*, and proceed on as before.

Now the *Quantity* of *Liquor* that would cover the *Crown* of the *Copper*, may be found without *Meaſuring* it, as above. In order to that, I do ſuppoſe the *Crown* to be the *Segment* of a *Sphere*, and the *Lower* Part of the *Copper* wherein the *Crown* ariſeth, to be the *Fruſtum* of a *parabolick Conoid*; then if the *Diameter* at the *Top* of the *Crown*, and its *perpendicular Height* are given, the *Quantity* of *Liquor* may be found by this following *Rule*:

Rule. *From the Area of the Plain at the Top of the Crown, Subſtract* 1 ⅓ *of the Area of the Crown's Height; the Remainder being Multiply'd into half the Height of the Crown, will produce the Quantity or Number of Gallons that will cover the Crown.*

This *Rule* is deduc'd from *Scholium*, *Page* 424, and *Theorem* 15. *Page* 430.

Sect. 5. *To compute the Content of any cloſe Cask in Gallons*, viz. *of any Butt, Pipe, Hogſhead, Barrel*, &c.

In order to perform this difficult Part of *Gauging*, the *Three* following *Dimenſions* of the propos'd *Cask* muſt be truly taken in *Inches*, and *Decimal* Parts of an *Inch*.

Viz.
- The *Bulge* or *Bung Diameter* within the *Cask*.
- Either of the *Head Diameters*, ſuppoſing them both *Equal*.
- And the Length of the *Cask* within.

Note, *In taking of theſe Dimenſions, it muſt be carefully obſerv'd*,

1. That the Bung-hole be in the middle of the Cask; alſo, that the Bung-ſtaff, and the Staff over-againſt the Bung-hole, are both regular or even within.

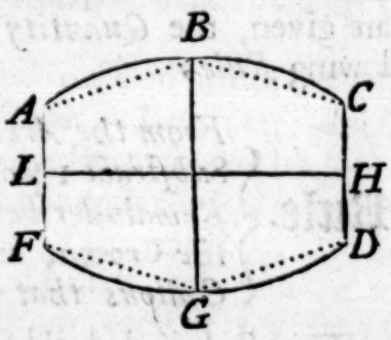
2. That the Heads of the Cask are equal and truly circular; if ſo, the Diſtance between the Inſide of the Chine to the Outſide of its oppoſite Staff, will be the Head Diameter within the Cask, very near.

3. With a ſliding Pair of Calipers, *(made on purpoſe for that Uſe)* take the ſhorteſt Diſtance at Length between the Outſides of the Two Heads; *(ſuppoſing them even)* from that Length ſubſtract 1 ½ Inch *(more, or leſs, according to the Largeneſs of the*

Cask) for the Thickneſs of the two Heads, the Remainder will be the Length of the Cask within.

Now, by theſe *Dimenſions*, one would ſuppoſe the *Content* of the *Cask* were perfectly limited; but it will be *eaſy* to perceive, by the following *Figure*, that the *Diameters* (*aboveſaid*) and the *Length* of one *Cask* may be *Equal* to thoſe of another, and yet one of thoſe *Casks* may contain or hold ſeveral *Gallons* more than the other.

As for Inſtance, ſuppoſe the *Annex'd Figure A B C D G F*, to repreſent a *Cask*; then it is plain, that if the *Outward curv'd Lines A B C*, and *F G D* are the Bounds or Staves of the Cask, it muſt needs hold more than if the *Inner Streight* or prick'd *Lines* were its Bounds or Staves; and yet the *Bung Diameter B G*, *Head Diameter C D* and *A F*, and the *Length L H* are the ſame in both thoſe *Casks*.



Whence it plainly appears, that no one *certain* or *general Rule* can be preſcrib'd to find the true *Content* of all Sorts of *Casks*, and therefore *Gaugers* do uſually ſuppoſe every *Cask* to be in the Form of ſome one of theſe following *Solids*.

Viz.
I. *The middle Zone or Fruſtum of a Spheroid.*
II. *The middle Zone or Fruſtum of a Parabolick Spindle.*
III. *The lower Fruſtums of two equal Parabolick Conoids.*
IV. *The lower Fruſtums of two equal Cones.*

Now the Way of *Gauging* at the *Cask*'s *Form*, and computing its *Content*, according to its ſuppos'd *Form*, I ſhall here ſhew in their Order.

I. If the *Staves* of the *Cask* are very *curved* or *Arching*, (*as the outward Lines of the laſt Figure*) then the *Cask* is ſuppos'd to be in the *Form* of the *middle Zone* or *Fruſtum* of *a Spheroid*, whoſe *Content* may be computed, by *Theorem* 22. *Page* 427. which gives theſe two *Rules*.

Rule 1. *To Twice the Square of the Bung Diameter add the Square of the Head Diameter; multiply that Sum into the Length, and divide the Product by* 1077,15. Viz. 3,8197 × 282 *for Ale Gallons; and by* 882,30. Viz. 3,8197 × 231 *for Wine Gallons. Or thus,*

Rule

Rule 2. *To Twice the Area of the Bung Circle, add the Area of the Head Circle; multiply their Sum into one Third of the Length, and the Product will be the Content in their reſpective Gallons.*

Example 1. Suppoſe a Cask in the Form of the *middle Zone* of a *Spheroid*, whoſe Bung Diameter is 31,5, Head Diameter 24,5, and its Length 42 *Inches*.

First 31,5 × 31,5 × 2 = 1984,5. And 24,5 × 24,5 = 600,25
Again 1984,5 + 600,25 = 2584,75. And 2584,75 × 42 = 108559,5
Then 1077,15) 108559,5 (100,78 the *Content* in *Ale Gallons*.
And 882,35) 108559,5 (123,03 the Content in *W. Gallons*.

Or thus, by the *Second Rule*.

Bung Diameter 31,5 *Twice* its *Circle's Area* is 5,5270
Head Diameter 24,5 its Circle's *Area* is 1,6718
The *Length* 42 *divided* by 3 is 14. 7,1988 = their *Sum*.
Then 7,1988 × 14 = 100,78 the Content in *A. Gallons*, as before
And ſo the Content in *Wine Gallons* may be found.

II. If the *Staves* of the Cask are not quite ſo much *curved* or *Arching*, as was ſuppos'd before, the Cask is then taken for the *middle Fruſtum* of a *parabolick Spindle*, and its Content is computed, as by *Theorem* 27. *Page* 432. Which gives this *Rule*.

Rule. *To Twice the Square of the* Bung Diameter, *add the Square of the* Head Diameter; *from their* Difference *Subſtract four Tenths of the Square of the* Difference *of the* Diameters; *multiply the Remainder into the Length, and divide the Product by* 1077,15, &c. *As above.*

Example 2. Suppoſe the Dimenſions the ſame as before. Then
31,5 × 31,5 × 2 : + 24,5 × 24,5 = 2584,75. And 31,5 − 24,5 = 7
Again 7 × 7 × 0,4 = 19,6. And 2584,75 − 16,6 : × 42 = 107736,3
Then 1077,15) 107736,3 (100,01 the *Cont.* in *A.G.*, &c. for *W.G.*

III. When the *Staves* of the Cask are but very little *Curved* or *Arching*, then it's ſuppos'd to be in the Form of the *Fruſtums* of Two equal *parabolick Conoids*, abutting or *joining* together upon one common *Baſe* at the *Bulge*, and the Content may be found by *Theorem* 25. *Page* 430. which gives theſe *Rules*.

Rule 1. *To the Square of the Bung Diameter add the Square of the Head Diameter; multiply their Sum into the Length, and Divide the Product by* 718,08 (viz. 2,5464 × 282) *for Ale Gallons; or by* 588,22 (viz. 2,5464 × 231) *for Wine Gallons. Or thus,*

Rule 2. *To the Area of the Bung Circle add the Area of the Head Circle; multiply the Sum into half the Length, and the Product will be the Content requir'd.*

Example 3. With the same *Dimensions* as before. Then 31,5×31,5 : + 24,5×24,5 = 1592,5. And 1592,5 × 42 = 66885
And 718,08) 66885 (93,01 the *Content* in *Ale Gallons.*
Or 588,22) 66885 (113,7 the *Content* in *Wine Gallons.*

IV. If the *Staves* of the *Cask* are *streight* from the *Bulge* to the *Head*, as the *Inner* prick'd *Lines* in the last *Figure*, *(if such a Cask can be made)* it is then taken for the *Lower Frustums* of Two equal *Cones*, abutting or *joining* together upon one common *Base* at the *Bulge*. And its *Content* may be computed as at *Problem* 13. *Page* 445. or by *Theorem* 15. *Page* 419. Thus,

Rule. *To the Sum of the Squares of the Head and Bung Diameters add their Product; then multiply that Sum into the Length, and Divide the last Product by* 1077,15. *Or by* 882,36. *The Quotient will be the Content, &c.*

Example 4. With the same *Dimensions* as before.

First 31,5 × 31,5 : + 24,5 × 24,5 : + 31,5 × 24,5 = 2364,25
And 2364,25 × 42 = 99298,5 Then 1077,15) 99298,5 (92,18 the *Content* in *Ale Gallons*. And so on for *Wine Gallons.*

Thus you have the *Methods* of computing the true *Contents* of the four *Solids*, in whose *Forms* all *Casks* are suppos'd to be. And by the *Examples* it appear, that *Four* such *Casks* as have their *Dimensions* all equal, and the same with those above-mention'd, their *Contents* will be as in the *Margin.*

	Ale Gallons.	*Differ.*
I.	100,78	
		0,77
II.	100,01	
		7,00
III.	93,01	
		0,83
V.	92,18	

From the *Disproportion* or *Inequality* of these *Differences* it will be *easy* to *conceive*, that there may be several *Casks* whose *Contens* cannot be truly found, according to the aforesaid suppos'd *Forms*; and therefore, in order to rectify the said *Inequalities*, some *Authors (that have written upon this Subject)* have laid down *Theorems* of their own *Invention*; *(and yet call'd them*

by

by these Names) others have propos'd *Tables* for the same Purpose. But since it is so, that we can only *guess* at the *Truth*, the *plainest* and *easiest* Way is to be preferr'd in *Practice*; and that is, by finding such a *mean Diameter* as will *reduce* the propos'd *Cask* to a *Cylinder*.

Thus, { *Multiply the Difference between the Head and Bung Diameters, with* 0,7. *or with* 0,65. *or with* 0,6. *or with* 0,55. *according as the Staves of the Cask are More or Less arching; Add the Product to the Head Diameter, and the Sum will be the mean Diameter requir'd. Then find the Content, as at Prob.* 11. *Page* 444.

Example. With the same *Dimensions* as before. Then the *Bung Diameter less*, the *Head Diam.* is 31,5 − 24,5 = 7. And

	M. D.	*A. G.*	*Cont.*	*Dif.*
24,5 + 7 × 0,7	= 29,4 its *Area*	2,4073 × 42	= 101,10	
24,5 + 7 × 0,65	= 29,05 ——	2,3504 × 42	= 98,71	2,39
24,5 + 7 × 0,6	= 28, 7 ——	2,2941 × 42	= 96,35	2,36
24,5 + 7 × 0,55	= 28,35 ——	2,2385 × 42	= 94,03	2,32

From these it may be observ'd, that the *Difference* between each *Cask*'s *Content* is *Regular*, and very near *Equal*; which plainly shews, that there is not so much Room left for *Error* this Way of *computing* their *Contents*, as was by the aforesaid *Forms*.

Now the *First* of these four (viz. *with* 0,7) is very commonly used amongst *Gauging* for all *Sorts* of *Casks*; but I did never *Gauge* any *Cask* that would *contain* quite so much as that *Rule* did make it; and the *Reason* doth appear very plain from *Theorem* 22. *Page* 427. being compar'd with *Theorem* 19. *Page* 426. and the last *Figure*, *viz.* that no *Cask* (*being regularly made*) can *hold more* than the *middle Frustum* of a *Spheroid*. But I always found by *Experience*, that if the *Second* and *Third* of these *Rules* (viz. *with* 0,65 *and* 0,6) were duly *Apply'd*, they would *answer* very near the *Truth* amongst the *common* Sort of *Casks*; and the *Fourth Rule* (viz *with* 0,55) will come pretty near the *Truth* in computing the *Contents* of *Casks*, whose *Staves* are almost *streight* betwixt the *Head* and *Bung*, *viz.* such as *Wine Pipes*, &c.

Sect 6. *To find what Quantity of Liquor is either Drawn forth, or Remaining in, any Spheroidal Cask, usually call'd the Ullage of a Cask, hath two Cases.*

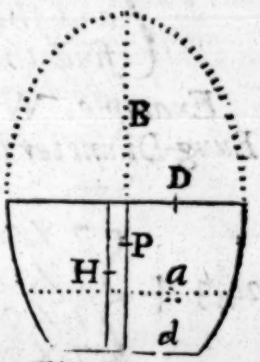
Case 1. *To find what Quantity of Liquor is in the Cask, when its Axis is perpendicular to the Horizon*, viz. *when it stands upright upon one of its Heads.*

In

In order to perform this the *eaſieſt* Way, it will be convenient to know how to *calculate* the *Area* of any *Circle* betwixt the *Bung* and *Head*, whoſe *Diſtance* from the *Bung* or *Middle* of the *Cask* is given. Now that may be done by this *Proportion*.

Viz. { *As the Square of half the Length of the Cask : is to the Difference between the Bung and Head Area's : : ſo is the Square of any Circle's Diſtance from the Bung : to the Difference between the Bung Area, and the Area of the Circle,* viz. *the Area of the Liquor's Surface.*

Demonſtration.



Let { H = *Half the Length of the Cask.* D = *Half the Bung Diameter.* d = *Half the Head Diameter.*

And { P = the *Diſtance* of any *Circle* from the *Bung.* a = *Half* the *Diameter* of that *Circle.*

Then according to the *common Property* of the *Ellipſis*, *Page* 368, it will be,

$BB : DD :: BB - HH : dd$. And $BB : DD :: BB - PP : aa$.

Ergo $\frac{DDHH}{DD - dd} = BB$. And $\frac{DDPP}{DD - aa} = BB$.

Conſequently, $\frac{DDHH}{DD - dd} = \frac{DD - aa}{DDPP}$

This *Æquation* being brought out of the *Fractions*, will become $DDHH - aaHH = DDPP - ddPP$.

Which gives this *Analogy* $HH : DD - dd :: PP : DD - aa$.

Then $DD - aa$ being *ſubſtracted* from DD, will leave aa. But *Circles Area's* are in *Proportion* to the *Squares* of their *Diameters*, by *Theorem* 6. *Page* 407. Therefore, *&c.* Q. E. D.

Then, from the *Bung Area* ſubſtract *one Third* Part of the aforeſaid *Difference*, viz. between the *Bung Area* and the *Area* of the *Liquor's Surface*; *multiply* the *Remainder* with the *Liquor's Diſtance* from the *Bung*, and the *Product* will ſhew what *Quantity* of *Liquor* is either *Above* or *Under half* the *Content* of the *Cask*.

Example. Let us ſuppoſe a *Cask* of the ſame *Dimenſions* with that in the *firſt Example*, *Page* 451. and let it be requir'd to find what *Quantity* of *Liquor* is in it, *(of Ale Meaſure)* when there is but 9 *Inches wet*. Here *half* the *Length* of the *Cask* is 21 Inches,

Inches, whoſe Square is 441, and the Liquor's Diſtance from the Bung is 21 — 9 = 12. Its Square is 144. The Difference between the *Bung*, and Head Area's is 1,0917 (= 2,7635 — 1,6718.) Then 441 : 1,0917 : : 144 : 0,3564.

And 2,7635 — 0,3564 = 2,4071 the *Area* of the *Liquor's Surface*.

Again 3) 0,3564 (0,1188. And 2,7635 — 0,1188 = 2,6447 Then 2,6447 × 12 = 31,7364, what the *Cask* wants of being *half full*. Conſequently 50,39 — 31,73 = 18,66 will be the *Quantity* of *Liquor* in the *Cask* at 9 *Inches wet* in *Ale Gallons*.

And if the *Cask* had wanted but 9 *Inches* of being *full*; then 50,39 + 31,73 = 82,12 would have been the *Quantity* of *Liquor* in the *Cask*.

Note, *Becauſe the Two Firſt Terms* (viz. 441 and 1,0917) *in the Proportion are fix'd*, viz. *continue the ſame for any Diſtance*, *'twill be very eaſy to calculate the Area's of all the Circles betwixt the Bung and Head to every Inch, and by that Means to make a Table that will ſhew what Quantity of Liquor is either drawn Out, or Remaining in the Cask, at any Depth.*

Caſe 2. *To find what Quantity of Liquor is in any Cask, when its Axis is Parallel to the Horizon*, viz. *when it lies along.*

There are Variety of *Tables* to be found in *Books* of *Gauging* for this Purpoſe; but I always obſerv'd, that the following *Method* of computing the *Ullage*, by a *Table* of the *Segments* of a *Circle*, came very near the *Truth* in all *Sorts* of *Casks*, which is thus perform'd:

1. By the *Bung* and *Head Diameters*, find ſuch a *mean Diameter* as you *judge* will *Reduce* the propos'd Cask to a Cylinder, by the *Method* laid down in *Page* 453. And then find its full *Content*, as in thoſe *Examples*.

2. From the *Bung Diameter* Subſtract the *mean Diameter*, and *half Difference*, (viz. *divide it by* 2.)

3. From the *Wet Inches* of the propos'd *Ullage*, Subſtract the ſaid *half Difference*, and call it *x*; then obſerve this *Proportion*.

Viz. { *As the Mean Diameter : is to* 100 (the Diameter of the Tabular Circle) : : *ſo is the laſt Difference* (viz. *x*) : *to a verſed Sine is the Table.* (Page 441.)

Then if the *Tabular Segment*, which ſtands againſt that *Verſed Sine*, be *multiply'd* into the Content of the Cask, the *Product* will ſhew the *Ullage*, viz. what *Quantity* of *Liquor* is either in the Cask, or *drawn* forth.

Example.

Example 1. Let the Cask be that of the *ſecond Sort*, in *Page* 453. *viz.* whoſe *Bung* Diameter is 31,5 *Inches*, mean Diameter 29,05 and Content 98,71 *Ale Gallons*; and ſuppoſe there were 10,5 *Inches Wet* in it, it is requir'd to find the *Wet*, and Dry *Gallons*?

Here 31,5—29,05=2,45 its *half* is 1,12. And 10,5—1,22=9,28
Then 29,05 : 100 : : 9,28 : 0,319=*V. Sine*; its *Segm.* is 0,2748
And 98,71 × 0,2748=27,12 the *Number* of *wet* Gallons.
Again 31,5—10,5 = 21 the dry Inches; and 21 — 1,22 = 19,78
Then 29,05 : 100 : : 19,78 : 0,68; its *Segment* is 0,7241
And 98,71 × 0,7241=71,48 the *Number* of dry Gallons.
Proof 71,48 + 27,12=98,6 the Contents of the Cask very near; which plainly ſhews the Truth of this Method.

Thus far may ſuffice concerning *Gauging* of Backs or Coolers, Tuns, Coppers and Casks, *&c.* To which I ſhall only add, That as the Contents of all *Brewers Utenſils* are to be computed by the Ale Gallons, ſo the Contents of all Diſtillers Utenſils (viz. all their Waſh-Backs, Stills, and Casks, *&c.*) muſt be computed by the Wine Gallon.

And in gauging of Malt (*upon which there is now a* Duty *of four* Shillings *per Buſhel*) you muſt obſerve, That a Corn or Malt-Buſhel doth contain 2150,42 *cubick Inches*; (See *Page* 42.) and therefore in gauging of Malt-Ciſterns, or other Veſſels, 2150,42 will be a *conſtant* or *fix'd* Diviſor for finding the *Area's* of Right-lin'd Figures in Buſhels at one Inch deep, and 2738 will be a conſtant or fix'd Diviſor for finding the Area's of Circular Figures.

I have omitted the Buſineſs of gauging Maſh-Tuns, and taken an Account of the Goods or Grains, in order to Eſtimate what Quantity of Worts were produc'd from them, *&c.* becauſe I could never find (*by* all *my Obſervations*) any *Certainty* therein; nor is it poſſible there ſhould be any, by Reaſon of the Great Difference that is in Malt, (*and its Grinding too*) for the beſt Malt (well *ground*) will yield or produce the moſt Worts, and leaſt Grains; on the contrary, bad Malt (*being* ill *ground*) yields the leaſt *Worts* and moſt *Grains*.

FINIS.